Model Matematika Numerik

Difusi Numeris

Djoko Luknanto
Departemen Teknik Sipil dan Lingkungan
Fakultas Teknik
Universitas Gadjah Mada

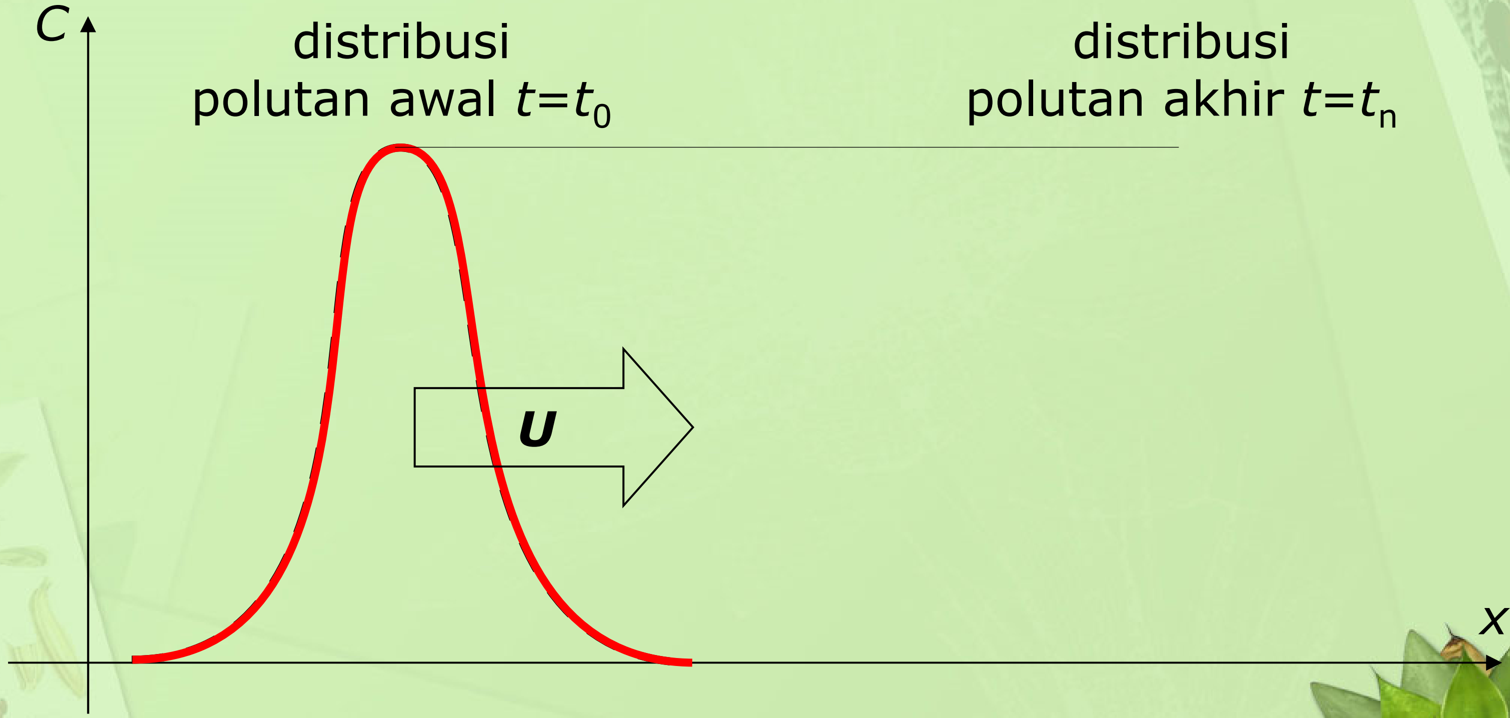
Persamaan Dasar

- Persamaan Adveksi Murni

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = 0$$

- *C*, konsentrasi polutan
  U, kecepatan aliran
  t, waktu
  x, ruang, lokasi

Interpretasi Persamaan Adveksi



Karena adveksi murni, maka distribusi polutan hanya bergerak karena pengaruh kecepatan aliran sebesar *U*, sedangkan bentuk distribusi konsentrasinya harus tetap.

Penyelesaian Karakteristik Linier

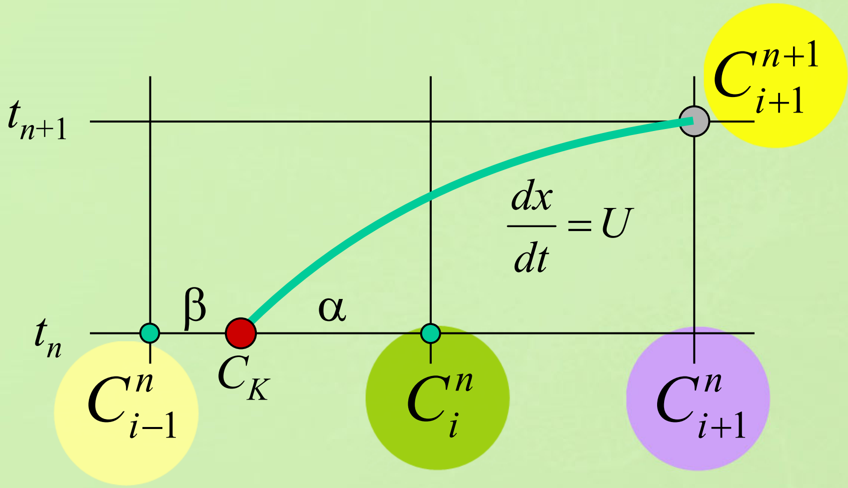
- Dengan metoda karakteristik dengan interpolasi linier diperoleh

$$C_{i+1}^{n+1} = \frac{\alpha}{\Delta x} C_{i-1}^{n} + \frac{\Delta x - \alpha}{\Delta x} C_i^n$$

akan diselidiki apakah formulasi diatas benar-benar merupakan penyelesaian persamaan adveksi murni:

$$\frac{\partial C}{\partial t} + U \frac{\partial C}{\partial x} = 0$$

Gunakan deret Taylor



- Ekspansikan semua parameter di atas:

C_{i-1}^{n} C_{i}^{n} C_{i+1}^{n+1}

ke parameter C_{i+1}^{n}

Deret Taylor

$$C_{i+1}^{n+1} \approx C_{i+1}^{n} + \frac{\partial C}{\partial t}\Delta t + \frac{\partial^2 C}{\partial t^2}\frac{\Delta t^2}{2}$$

$$C_{i}^{n} \approx C_{i+1}^{n} - \frac{\partial C}{\partial x}\Delta x + \frac{\partial^2 C}{\partial x^2}\frac{\Delta x^2}{2}$$

$$C_{i-1}^{n} \approx C_{i+1}^{n} - \frac{\partial C}{\partial x}2\Delta x + \frac{\partial^2 C}{\partial x^2}2\Delta x^2$$

- Faktor turunan kedua dari *C* terhadap *t* dijelaskan pada slide berikut.

Manipulasi Pers. Dasar

- Pers. Dasar diubah menjadi

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = 0 \Rightarrow \frac{\partial C}{\partial t} = -U\frac{\partial C}{\partial x}$$

$$\frac{\partial^2 C}{\partial t^2} = \frac{\partial}{\partial t}\left(-U\frac{\partial C}{\partial x}\right) \Rightarrow \frac{\partial^2 C}{\partial t^2} = -U\frac{\partial}{\partial t}\left(\frac{\partial C}{\partial x}\right)$$

- Jadi

$$\frac{\partial^2 C}{\partial t^2} = -U\frac{\partial}{\partial x}\left(\frac{\partial C}{\partial t}\right) \Rightarrow \frac{\partial^2 C}{\partial t^2} = -U\frac{\partial}{\partial x}\left(-U\frac{\partial C}{\partial x}\right)$$

$$\frac{\partial^2 C}{\partial t^2} = U^2\frac{\partial^2 C}{\partial x^2}$$

Diperoleh korelasi

$$C_{i+1}^{n+1} \approx C_{i+1}^{n} + \frac{\partial C}{\partial t}\Delta t + U^2 \frac{\partial^2 C}{\partial x^2}\frac{\Delta t^2}{2}$$

$$C_{i}^{n} \approx C_{i+1}^{n} - \frac{\partial C}{\partial x}\Delta x + \frac{\partial^2 C}{\partial x^2}\frac{\Delta x^2}{2}$$

$$C_{i-1}^{n} \approx C_{i+1}^{n} - \frac{\partial C}{\partial x}2\Delta x + \frac{\partial^2 C}{\partial x^2}2\Delta x^2$$

... manipulasi lanjut ... 1

$$C_{i+1}^{n+1} = \frac{\alpha}{\Delta x} C_{i-1}^{n} + \frac{\Delta x - \alpha}{\Delta x} C_{i}^{n}$$

dengan

$$C_{i+1}^{n+1} \approx C_{i+1}^{n} + \frac{\partial C}{\partial t}\Delta t + U^2 \frac{\partial^2 C}{\partial x^2}\frac{\Delta t^2}{2}$$

$$(1-\frac{\alpha}{\Delta x})C_{i}^{n} \approx (1-\frac{\alpha}{\Delta x})C_{i+1}^{n} - (1-\frac{\alpha}{\Delta x})\frac{\partial C}{\partial x}\Delta x + (1-\frac{\alpha}{\Delta x})\frac{\partial^2 C}{\partial x^2}\frac{\Delta x^2}{2}$$

$$\frac{\alpha}{\Delta x}C_{i-1}^{n} \approx \frac{\alpha}{\Delta x}C_{i+1}^{n} - \frac{\alpha}{\Delta x}\frac{\partial C}{\partial x}2\Delta x + \frac{\alpha}{\Delta x}\frac{\partial^2 C}{\partial x^2}2\Delta x^2$$

... manipulasi lanjut ... 2

$$\frac{\partial C}{\partial t}\Delta t + U^2\frac{\partial^2 C}{\partial x^2}\frac{\Delta t^2}{2} = -(1+\frac{\alpha}{\Delta x})\frac{\partial C}{\partial x}\Delta x + (1+3\frac{\alpha}{\Delta x})\frac{\partial^2 C}{\partial x^2}\frac{\Delta x^2}{2}$$

$$\frac{\partial C}{\partial t}\Delta t + (1+\frac{\alpha}{\Delta x})\frac{\partial C}{\partial x}\Delta x = (1+3\frac{\alpha}{\Delta x})\frac{\partial^2 C}{\partial x^2}\frac{\Delta x^2}{2} - U^2\frac{\partial^2 C}{\partial x^2}\frac{\Delta t^2}{2}$$

$$\frac{\partial C}{\partial t}\Delta t + (\Delta x+\alpha)\frac{\partial C}{\partial x} = (1+3\frac{\alpha}{\Delta x})\frac{\partial^2 C}{\partial x^2}\frac{\Delta x^2}{2} - U^2\frac{\partial^2 C}{\partial x^2}\frac{\Delta t^2}{2}$$

$$\left(\frac{\partial C}{\partial t} + \frac{\Delta x+\alpha}{\Delta t}\frac{\partial C}{\partial x}\right)\Delta t = (1+3\frac{\alpha}{\Delta x})\frac{\partial^2 C}{\partial x^2}\frac{\Delta x^2}{2} - U^2\frac{\partial^2 C}{\partial x^2}\frac{\Delta t^2}{2}$$

... manipulasi lanjut ... 3

substitusi (lihat gambar di depan): $U\Delta t = \Delta x + \alpha$

$$\left(\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x}\right)\Delta t = (1 + 3\frac{\alpha}{\Delta x})\frac{\partial^2 C}{\partial x^2}\frac{\Delta x^2}{2} - \frac{\partial^2 C}{\partial x^2}\frac{\Delta x^2 + 2\alpha\Delta x + \alpha^2}{2}$$

$$\left(\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x}\right)\Delta t = (1 + 3\frac{\alpha}{\Delta x})\frac{\partial^2 C}{\partial x^2}\frac{\Delta x^2}{2} - \frac{\partial^2 C}{\partial x^2}\frac{\Delta x^2 + 2\alpha\Delta x + \alpha^2}{2}$$

$$\left(\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x}\right)\Delta t = \cancel{\frac{\Delta x}{2}\frac{\partial^2 C}{\partial x^2}} + \frac{3}{2}\alpha\Delta x\frac{\partial^2 C}{\partial x^2} - \cancel{\frac{\Delta x^2}{2}\frac{\partial^2 C}{\partial x^2}} - \alpha\Delta x\frac{\partial^2 C}{\partial x^2} - \frac{\alpha^2}{2}\frac{\partial^2 C}{\partial x^2}$$

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = \alpha\left(\frac{\Delta x - \alpha}{2\Delta t}\right)\frac{\partial^2 C}{\partial x^2}$$

… manipulasi lanjut … 4

Definisi: $Cr = \dfrac{U\Delta t}{\Delta x}$ $\quad U\Delta t = \Delta x + \alpha$

$$\alpha = U\Delta t - \Delta x = Cr\Delta x - \Delta x = (Cr-1)\Delta x$$

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = \alpha\left(\frac{\Delta x - \alpha}{2\Delta t}\right)\frac{\partial^2 C}{\partial x^2}$$

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = \frac{\Delta x^2}{2\Delta t}(Cr-1)(2-Cr)\frac{\partial^2 C}{\partial x^2}$$

... manipulasi lanjut ... 5

Definisi lebih umum: $Cr = \dfrac{U\Delta t}{\Delta x}$ $\quad U\Delta t = k\Delta x + \alpha$

$$\alpha = U\Delta t - k\Delta x = Cr\Delta x - k\Delta x = (Cr - k)\Delta x$$

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = \alpha\left(\frac{\Delta x - \alpha}{2\Delta t}\right)\frac{\partial^2 C}{\partial x^2}$$

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = \underbrace{(Cr - k)(1 - Cr + k)\frac{\Delta x^2}{2\Delta t}}_{K_n}\frac{\partial^2 C}{\partial x^2}$$

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = K_n\frac{\partial^2 C}{\partial x^2}$$

Persamaan Dasar Berubah

- Dengan metoda karakteristik (linier) ini persamaan berubah menjadi

$$\frac{\partial C}{\partial t} + U \frac{\partial C}{\partial x} = K_n \frac{\partial^2 C}{\partial x^2}$$

K_n disebut difusi numeris, karena koefisien tersebut merupakan *side-effect* dari teknik numerik yang dipilih.

Koefisien Difusi Numeris

- Koefisien Difusi Numeris, K_n merupakan fungsi dari bilangan Courant:

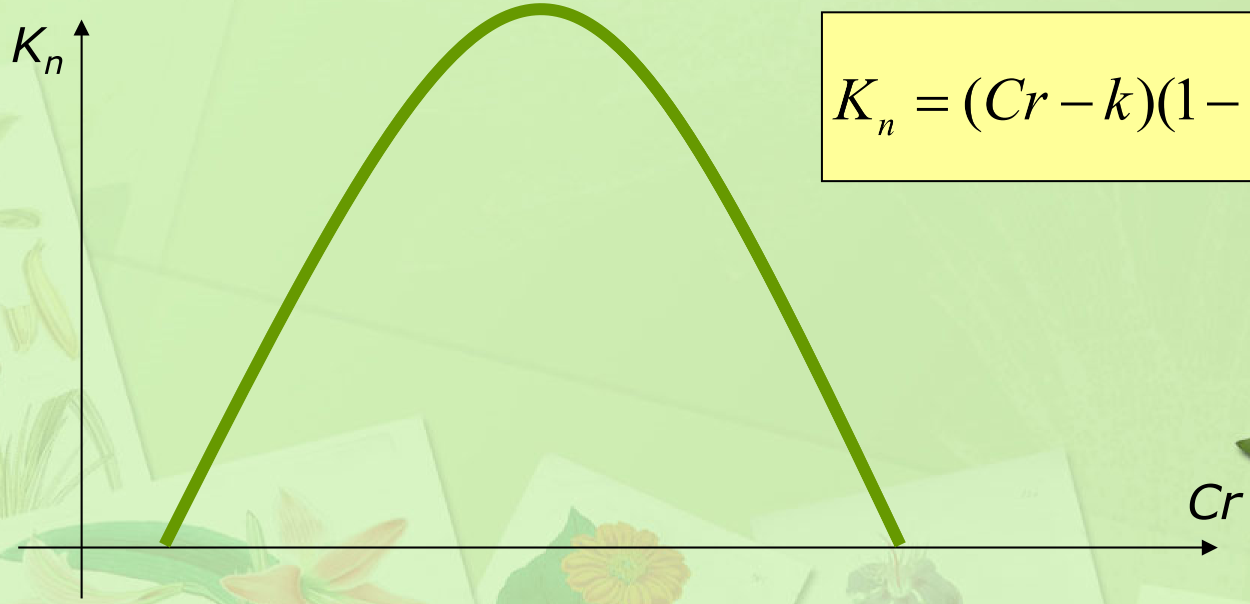
$$Cr = \frac{U\Delta t}{\Delta x}$$

K_n versus *Cr*

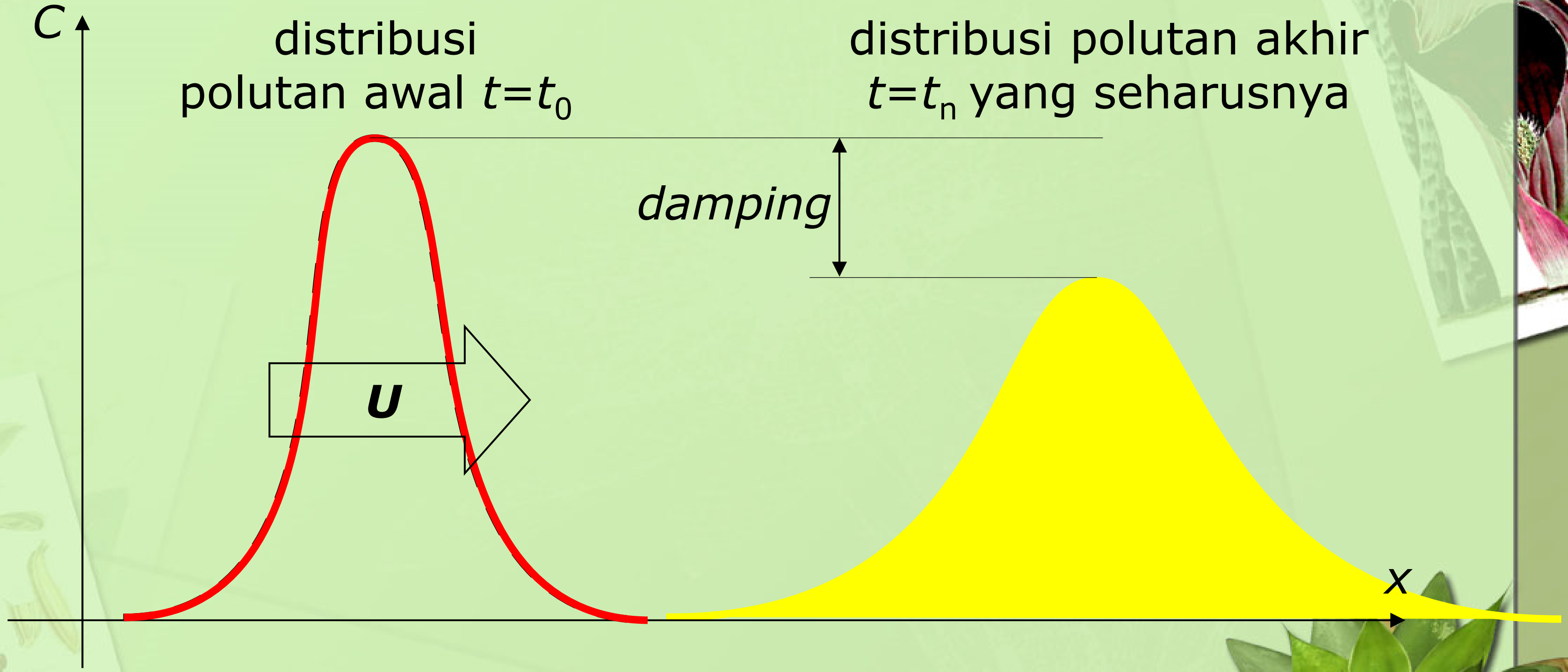
- Contoh kasus korelasi antara K_n versus *Cr*

- K_n membesar maka penyelesaiannya mengalami *damping* makin besar.

$$K_n = (Cr - k)(1 - Cr + k)\frac{\Delta x^2}{2\Delta t}$$



Difusi Numerik vs *Damping*



Karena Difusi Numeris, K_n, maka distribusi polutan tidak hanya bergerak karena pengaruh kecepatan aliran sebesar *U*, namun mengalami dispersi semu karena teknik numerik yang digunakan.

Skema-skema lain

- Untuk skema-skema beda hingga yang lain misalkan skema maju dan skema mundur dapat dilakukan hal yang sama dan dapat diperoleh sifat difusi numerisnya.
- Demikian pula untuk persamaan dasar yang lainnya dapat diperoleh karakteristik penyelesaian numeriknya terhadap skema yang digunakan.