III. TEORI INTERPOLASI

Jika kita mempunyai satu set data :

$(x_0, y_0), (x_1, y_1), \qquad , (x_n, y_n)$

maka dalam bab ini akan dijelaskan bagaimana cara mencari polinomial yang melalui data di atas.

Jika polinomial ini ditulis sebagai :

$$p(x) = a_0 + a_1 x + \qquad + a_n x^n$$

maka jika data di atas disubstitusikan akan didapat $(n+1)$ persamaan dengan $(n+1)$ bilangan anu yaitu $a_0, \quad , a_n$

$$\begin{array}{l} a_0 + a_1 x_0 + \quad \cdots \quad + a_n x_0^n = y_0 \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ a_0 + a_1 x_n + \quad \cdots \quad + a_n x_n^n = y_n \end{array}$$

Persamaan di atas jika diselesaikan akan menghasilkan $a_0, \quad , a_n$ sehingga polinomial $p(x)$ dpt dicari.

III.1. Metoda Beda Terbagi Newton

Notasi yang digunakan :

$$f[x_0, x_1, \quad , x_n] = \frac{f[x_1, \quad , x_n] - f[x_0, \quad , x_{n-1}]}{x_n - x_0}$$

Contoh :

Order 0 : $\quad f[x_0] = f(x_0)$

Order 1 : $f[x_0, x_1] = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0}$

Order 2 : $f[x_0, x_1, x_2] = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$

Order 3 : $f[x_0, x_1, x_2, x_3] = \dfrac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$

Rumus Beda Terbagi Newton :

$$p_n(x) = f[x_0] + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2]$$
$$+ \cdots + (x - x_0) \ldots (x - x_{n-1}) f[x_0, x_1, \quad , x_n]$$

Contoh : Kita buat tabel beda terbagi berdasarkan
$f(x) = x^3 - 2x^2 + 7x - 5$

i	x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f_2[\ ]$	$f_3[\ ]$
0	0.	-5.			
			6. A		
1	1.	1.		2. D	
			12. B		1. F
2	3.	25.		6. E	
			30. C		
3	4.	55.			

Keterangan : $A = \dfrac{1 - (-5)}{1 - 0} = 6$, $B = \dfrac{25 - 1}{3 - 1} = 12$

$C = \dfrac{55 - 25}{4 - 3} = 30$, $D = \dfrac{B - A}{3 - 0} = 2$

$E = \dfrac{30 - 12}{4 - 1} = 6$, $F = \dfrac{E - D}{4 - 0} = 1$

Contoh hitungan $p_n(x = 0.5) = ?$

- $p_1(x) = -5 + (x-0)6 = 6x - 5$

  $\therefore p_1(0.5) = -2$

- $p_2(x) = -5 + (x-0)6 + (x-0)(x-1)2 = 2x^2 + 4x - 5$

  $\therefore p_2(0.5) = -2.5$

- $p_3(x) = -5 + (x-0)6 + (x-0)(x-1)2 + (x-0)(x-1)(x-3)1$
  $= x^3 - 2x^2 + 7x - 5$

  $\therefore p_3(0.5) = -15/8 = -1.875$

- Algoritma metoda beda terbagi Newton.

Divdif (d, x, n)

1. Keterangan : d dan x adalah vektor $f(x_i)$ dan x_i $i = 0, 1, \ldots, n$. Pada saat 'exit' d_i akan terisi oleh nilai $f[x_0, \ldots, x_i]$

2. Kerjakan s/d langkah 4 untuk $i = 1, 2, \ldots, n$

3. Kerjakan s/d langkah 4 untuk $j = n, n-1, \ldots, i$

4. $d_j := (d_j - d_{j-1})/(x_j - x_{j-i})$

5. 'exit'

Interp (d, x, t, p)

1. Keterangan : Pada awalnya d dan x adalah vektor dari $f[x_0, \quad , x_i]$ dan x_i, $i = 0, 1, \quad , n$. Pada saat 'exit' p akan berisi $p_n(t)$

2. $p := d_n$

3. Kerjakan s/d langkah 4 utk $i = n-1, n-2, \quad , 0$

4. $p := d_i + (t - x_i)p$

5. 'exit'

III.2. Interpolasi dgn beda hingga bertabel

- Beda maju

Notasi : $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$

dimana $x_i = x_0 + ih$, $i = 0, 1, 2, 3, \ldots$

Untuk $r \geqslant 0$,

$$\Delta^{r+1} f(z) = \Delta^r f(z+h) - \Delta^r f(z)$$

dimana $\Delta^0 f(z) = f(z)$

$\Delta^r f(z)$ disebut 'beda maju order r'

Δ disebut 'operator beda maju'

Contoh : $\Delta^0 f(x) = f(x)$

$$\begin{aligned}\Delta f(x) &= \Delta^0 f(x+h) - \Delta^0 f(x)\\ &= f(x+h) - f(x)\end{aligned}$$

$$\Delta^2 f(x) = \Delta f(x+h) - \Delta f(x)$$

Contoh hitungan : Kita pakai polinomial
$x^3 - 2x^2 + 7x - 5$ dgn $h = 1.0$

i	x_i	$f(x_i)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
0	0.	-5.				
			6.			
1	1.	1.		2.		
			8.		6.	
2	2.	9.		8.		0.
			16.		6.	
3	3.	25.		14.		
			30.			
4	4.	55.				

Korelasi antara 'beda maju' dgn 'beda terbagi' :

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0 + h) - f(x_0)}{h} = \frac{\Delta f(x_0)}{h}$$

$$f[x_0, x_1, x_2] = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

$$= \frac{\Delta^2 f(x_0)}{2h^2}$$

Secara umum : $f[x_0, x_1, \quad , x_n] = \dfrac{\Delta^n f(x_0)}{n!\, h^n}$

- Akan dijabarkan rumus interpolasi 'beda maju' dari rumus 'interpolasi 'beda terbagi' Newton

Didefinisikan $\alpha = \dfrac{x - x_0}{h}$ yang menunjukkan letak titik x terhadap x_0. Jadi misalnya $\alpha = 1.6$, maka x terletak pada jarak 6/10 dari x_1 kearah x_2.

Diinginkan rumus utk

$$(x-x_0)(x-x_1)\ldots\ldots(x-x_k)$$

dinyatakan dalam α

$$x - x_j = x_0 + \alpha h - (x_0 + jh) = (\alpha - j)h$$

Jadi $(x-x_0)\ldots(x-x_k) = \alpha(\alpha-1)\ldots\ldots(\alpha-k)h^{k+1}$

Sehingga

$$p_n(x) = f_0 + \alpha h \frac{\Delta f_0}{h} + \alpha(\alpha-1)h^2 \frac{\Delta^2 f_0}{2!\,h^2} + \ldots\ldots$$
$$+ \alpha(\alpha-1)\ldots\ldots(\alpha-n+1)h^n \frac{\Delta^n f_0}{n!\,h^n}$$

Jika didefinisikan koefisien binomial sbb :

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!} \qquad k > 0$$

dan

$$\binom{\alpha}{0} = 1$$

maka didapat rumus interpolasi 'beda maju' sbb

$$p_n(x) = \sum_{j=0}^{n} \binom{\alpha}{j} \Delta^j f(x_0)$$

dimana $\alpha = \dfrac{x - x_0}{h}$

Contoh hitungan : $p_n(x = 1.5) = ?$

$$\alpha = \frac{x - x_0}{h} = \frac{1.5 - 0.}{1.} = 1.5$$

- $p_1(x) = f(x_0) + \alpha \Delta f(x_0)$

  $= -5 + 1.5(6) = 4$

- $p_2(x) = f(x_0) + \alpha \Delta f(x_0) + \dfrac{\alpha(\alpha - 1)}{2!} \Delta^2 f(x_0)$

  $= -5 + 1.5(6) + 1.5(0.5)\dfrac{2}{2!} = 4.75$

Beda mundur

Notasi : $\nabla f(z) = f(z) - f(z-h)$

$$\nabla^{r+1} f(z) = \nabla^r f(z) - \nabla^r f(z-h) \qquad r \geq 1$$

Rumus interpolasinya

$$p_n(x) = \sum_{j=0}^{n} \binom{j-1-\alpha}{j} \nabla^j f(x_0)$$

dimana $\alpha = \dfrac{x_0 - x}{h}$, $\binom{-1-\alpha}{0} = 1$

i	x_i	$f(x_i)$	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
-4	0.	-5.				
			6.			
-3	1.	1.		2.		
			8.		6.	
-2	2.	9.		8.		0.
			16.		6.	
-1	3.	25.		14.		
			30.			
0	4.	55.				

Contoh $p_n(x = 3.5) = ?$

$$\alpha = \frac{x_0 - x}{h} = \frac{-3.5+4.}{1.0} = +0.5$$

- $p_1(x) = f(x_0) + (-\alpha)\nabla f(x_0)$

  $p_1(3.5) = 55 + (-0.5)\,30 = 40$

- $p_2(x) = p_1(x) + (-\alpha)(-\alpha+1)\,\nabla^2 f(x_0)/2!$

  $p_2(3.5) = 40 + (-0.5)(0.5)\,14/2! = 38.25$

- $p_3(x) = p_2(x) + (-\alpha)(-\alpha+1)(-\alpha+2)\,\nabla^3 f(x_0)/3!$

  $p_3(3.5) = 38.25 + (-0.5)(0.5)(1.5)\,6/3! = 37.875$

Lagrange

$$p_n(x) = \sum_{i=0}^{n} L_i(x)\, f(x_i)$$

$$L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x-x_j}{x_i-x_j} \qquad i = 0,1, \quad ,n$$

Contoh :

$$p_1(x) = \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)$$

$$p_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Contoh : hitung $p_2(x)$ yang melalui titik-titik $(0,-5)$; $(1,1)$; $(3,25)$

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{x^2-4x+3}{3}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-3)}{(1-0)(1-3)} = \frac{x^2-3x}{-2}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{x^2-x}{6}$$

Jadi $p_2(x) = L_0(x)\times(-5) + L_1(x)\times(1) + L_2(x)\times(25)$
$= 2x^2+4x-5$

- Beberapa fakta penting dari 'beda terbagi'

1) $f[x_0, x_1, \quad , x_m] = \dfrac{f^{(m)}(\xi)}{m!}$ utk $\xi \in \mathcal{H}\{x_0, x_1, \quad x_m\}$

dimana $\mathcal{H}\{x_0, \quad , x_m\}$ artinya interval terkecil dimana $x_0, x_1, \quad , x_m$ tercakup!

Contoh : $f[x_0] = \dfrac{f^{(0)}(\xi)}{0!} = f(x_0)$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(\xi) \qquad \xi \in [x_0, x_1]$$

$$f[x_0, x_1, x_2] = \tfrac{1}{2} f''(\xi) \qquad \xi \in \mathcal{H}\{x_0, x_1, x_2\}$$

2) Jika $f(x)$ adalah polinomial derajad m, maka

$$f[x_0, \quad , x_n, x] = \begin{cases} \text{polinomial derajad } m-n-1 & n < m-1 \\ a_m & n = m-1 \\ 0 & n > m-1 \end{cases}$$

dimana $f(x) = a_m x^m + a_{m-1} x^{m-1} + \qquad + a_1 x + a_0$

3) Kesalahan dalam interpolasi.

$$f(x) - p_n(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(n+1)!} f^{(n+1)}(\xi_x)$$

dimana $\xi_x \in \mathcal{H}\{x_0, \quad , x_n, x\}$

4) $\dfrac{d}{dx} f[x_0, \ldots, x_n, x] = f[x_0, x_1, \quad , x_n, x, x]$