Magic squares from simple squares and a method like Conways LUX

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1 author:

Hariprasad Manjunath
Indian Institute of Science

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M Hariprasad

Department of Computational and Data Sciences,
Indian Institute of Science, Bangalore-560012, India
e-mail: mhariprasadkansur@gmail.com

Abstract: This article studies the duality between simple square arrangement of numbers and magic squares. For even ordered simple square, corresponding topological arrangement arising from cycles and reverse cycles is discussed. A Conways LUX like construction method is presented for singly even magic square. The article concludes with few observational remarks on the duality property between magic squares and simple square arrangements.
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1 Introduction and notations

A normal magic square is an arrangement of natural numbers from $1, 2, \cdots, n^2$ into a square, such that every row, every columns and diagonals have the same sum. The sum is called magic sum and it is equal to $n\frac{n^2+1}{2}$. There are several methods for constructing the magic squares which involve following an elementary pattern, adjoining two or more elementary squares and so on. One such method is well known for constructing the odd dimensional magic squares which follow diagonal filling of the square, this was proposed by Simon de la Loubere [1] also known as Siamese method [3]. In this article we discuss the relation between the simple square arrangement of numbers from one to n^2 and the odd magic square obtained by Siamese method. Also the analysis shows why the method is not applicable for constructing even dimensional magic square. Further we present a construction method for singly even magic square.

Let I_n be the identity matrix of dimension $n \times n$. Let C be the adjacency matrix of a directed cycle, denoted by $C = \begin{bmatrix} 0 & 1 \\ I_{n-1} & 0 \end{bmatrix}$. This matrix can also be described by the relation $i - j = 1 \mod n$ having one for the i, j entry satisfying the relation and zero otherwise. Let J be the left to right column flip of identity matrix, denoted by the relation $i = -j + 1 \mod n$. A be the square arrangement of the integers from 1 to n^2, having entry $A(i, j) = (i-1)n+j$ where $1 \leq i, j \leq n$. The matrix CJ is denoted by the relation $i + j = 2 \mod n$. Let $A.B$ denote the hadamard product of two matrices A and B. The non zero entris of $A.C^j$, are called entries of j^{th} cycle of A. Similarly the non zero entries of $A.(C^i J)$ are called the entries of i^{th} reverse cycle of A.

2 Results

Lemma 2.1. *The sum of entries of square arrangement A along any cycles and along any reverse cycles is equal to the magic sum.*

Proof. Any cycle is defined by the relation $i - j = l \mod n$ for $1 \leq l \leq n$. So we get,

$$\sum_{i-j=l \mod n} A(i,j) = \sum_{i-j=l \mod n} (i-1)n + j. \tag{1}$$

For every i we have a corresponding j in the relation, so when i varies among 1 to n, we have j also varying fron 1 to n. So the summation can be split,

$$\sum_{i-j=l \mod n} A(i,j) = \sum_{i=1}^{n}(i-1)n + \sum_{j=1}^{n} j, \tag{2}$$

$$= \frac{n^2(n+1)}{2} - n^2 + \frac{n(n+1)}{2}, \tag{3}$$

$$= \frac{n(n^2+1)}{2}. \tag{4}$$

Similarly the proof can be carried out for reverse cycles. □

Also note that the sum of the central row and central column entries of odd dimensional square arrangement is equal to the magic sum. The central column sum of the matrix A is given by,

$$\sum_{k=0}^{n-1} \frac{n+1}{2} + kn, \tag{5}$$

$$= n\frac{n+1+(n-1)(n)}{2}, \tag{6}$$

$$= \frac{n(n^2+1)}{2}. \tag{7}$$

Similarly the central row sum in the matrix A is given by,

$$\sum_{k=1}^{n} \frac{n-1}{2} n + k, \tag{8}$$
$$= n \frac{n(n-1) + n + 1}{2}, \tag{9}$$
$$= \frac{n(n^2+1)}{2}. \tag{10}$$

2.1 Row-Column systems

Now we axiomatize rows and columns of a square arrangement,

1. There are n rows in the square arrangement each having n elements, no two rows intersect each other.

2. There are n non intersecting columns in the square arrangement each having n elements and every column intersects every row exactly once.

From a set of n^2 elements if we are able to identify such n rows and n columns, then it is possible to put them in a square arrangement. By specifying the order of rows and order of columns uniquely specifies the square arrangement hence defining the diagonals.

Now we verify that the properties 1 and 2 are satisfied by n cycles and n reverse cycles of the square arrangent A, when n is odd.
Property 1 : It is easy to see that C^k and C^l do not intersect, as $i - j = l \mod n$ uniquely determines i given j. Each C^i has n nonzero elements.
Property 2 : There are n reverse cycles. The relation $i + j = l \mod n$ uniquely determines j given i. To see the intersection of the rows and column, we need i, j satisfying

$$i - j = l \mod n, \tag{11}$$
$$i + j = m \mod n. \tag{12}$$

For solving this, we get

$$2i = l + m \mod n, \tag{13}$$
$$2j = m - l \mod n \tag{14}$$

This system has unique solution when modulo inverse of 2 is defined with respect to n. This is possible for odd n. So every row (cycle) intersect every column

(reverse cycle) exactly once, when n is odd.
However when n is even, we have two cases,
l and m both even or both odd : in this case we have two solutions for i and j. Which are $\left(\frac{l+m}{2}, \frac{l-m}{2}\right)$ and $\left(\frac{l+m\pm n}{2}, \frac{l-m\pm n}{2}\right)$.
l is even m is odd or m is even l is odd : Then the system is not solvable.

Hence, for odd n, the square arrangement can be rearranged to obtain the magic square. This can be done by looking at diagonals in the square lattice arrangement by A. By looking at the diagonal arrangement, the linear rows and columns of the matrix A will be reshaped into cycles and reverse cycles whereas the cycles and reverse cycles will be reshaped into rows and columns.

2.2 Duality

Let (R, L) and (S, T) be two row column systems. When a square A is reshaped into a square B by reordering the elements in R_i (and L_j) of A into S_i (and T_j) of B. If this reordering also maps S_i (and T_j) of A into R_i(and L_j) of B for $1 \leq i, j \leq n$, then we say A and B have dual row column systems.

Let $vec(A)$ be the $n^2 \times 1$ vector obtained by appending all the rows of A in a single column. Let P be the permutation matrix which maps $vec(A) \to vec(M)$.

Theorem 2.1. *The two row column systems are dual of each other if and only if $P^2 = q \otimes s$ for order n permutation matrices q and s.*

Proof. Since P maps simple square to the magic square, in order for the row column system to be dual, it should map magic square to the permuted simple square. Thus we have,

$$Pvec(A) = vec(M), \tag{15}$$
$$Pvec(M) = vec(\tilde{A}), \tag{16}$$
$$P^2 vec(A) = vec(\tilde{A}) \tag{17}$$

Here the vector $vec\tilde{A}$ has permutation of row elemnts due to column rearrangements and permutation of row positions. All the row permutations must be same because otherwise result into different column elements than the simple square. Thus if two row column systems are dual, then $P^2 = q \otimes s$. □

As an example the 3×3 simple square

1	2	3
4	5	6
7	8	9

is mapped to magic square

8	1	6
3	5	7
4	9	2

via a matrix P by the relation,

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 6 \\ 3 \\ 5 \\ 7 \\ 4 \\ 9 \\ 2 \end{bmatrix}.$$

The matrix P satisfies $P^2 = J \otimes J$ with J being left to right flip of identity matrix.

Consider the 5 by 5 arrangement ,

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

. The corresponding lattice arrangement is given by,

1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
6	7	8	9	10	6	7	8	9	10	6	7	8	9	10
11	12	13	14	15	11	12	13	14	15	11	12	13	14	15
16	17	18	19	20	16	17	18	19	20	16	17	18	19	20
21	22	23	24	25	21	22	23	24	25	21	22	23	24	25
1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
6	7	8	9	10	6	7	8	9	10	6	7	8	9	10
11	12	13	14	15	11	12	13	14	15	11	12	13	14	15
16	17	18	19	20	16	17	18	19	20	16	17	18	19	20
21	22	23	24	25	21	22	23	24	25	21	22	23	24	25
1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
6	7	8	9	10	6	7	8	9	10	6	7	8	9	10
11	12	13	14	15	11	12	13	14	15	11	12	13	14	15
16	17	18	19	20	16	17	18	19	20	16	17	18	19	20
21	22	23	24	25	21	22	23	24	25	21	22	23	24	25

.

By looking at the diagonals in the green cells, corresponding magic square is given by

14	10	1	22	18
20	11	7	3	24
21	17	13	9	5
2	23	19	15	6
8	4	25	16	12

. Note that cycles and reverse cycles of this magic square are rows and columns of 5 by 5 simple square arrangement. Magic squares formed by green cells and pink cells together cover the lattice.

One can also look at the lattice formed by A^T, for example,

1	4	7
2	5	8
3	6	9

.

By arranging this into a lattice we get,

1	4	7	1	4	7	1	4	7
2	5	8	2	5	8	2	5	8
3	6	9	3	6	9	3	6	9
1	4	7	1	4	7	1	4	7
2	5	8	2	5	8	2	5	8
3	6	9	3	6	9	3	6	9
1	4	7	1	4	7	1	4	7
2	5	8	2	5	8	2	5	8
3	6	9	3	6	9	3	6	9

.

The entries colored in the green are given by

8	1	6
3	5	7
4	9	2

, which is an order 3 magic square.

In case of even n, cycles and reverse cycles fail to form rows and columns. but we obtain the following type of arrangement by listing the even and odd cycles seperately. Consider the arrangement from order 4 square,

4	10	5	15
13	7	2	12
3	9	8	14
6	16	11	1

.

Here every row sum is magic sum and every coloured square sum is magic sum, but every column sum is not the magic sum.

Similarly from the order 6 simple square arrangement, by listing even cycles and odd cycles seperately, we get

6	21	14	35	28	7
11	26	19	4	18	33
16	31	9	30	23	2
5	20	12	27	13	34
10	25	17	32	3	24
15	36	22	1	8	29

.

2.3 A LUX like method to construct singly even magic square

The method of constructing odd magic square can be used for constructing magic square when n is a singly even number, i.e. it is twice an odd integer. Conways LUX method is one such method [3]. We dedicate this section to John H Conway. Let A be a matrix denoting the simple square. Then A can be looked as the odd square of 2×2 blocks denoted by $A_2(i)$ for $1 \leq i \leq \left(\frac{n}{2}\right)^2$. Let A_2 be the matrix obtained by rearranging the blocks $A_2(i)$ according to the position of i in the Siamese magic square of order $\left(\frac{n}{2}\right)$. Then we can see that sum of pairs of columns, rows and diagonals (corresponding to the blocks) in A_2 will be twice the magic sum. We call the matrix A_2 as the almost magic square.

For a simple square of order six,

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

.

By re-arranging the 2×2 blocks according to 3×3 magic square

8	1	6
3	5	7
4	9	2

we get,

27	28	1	2	17	18
33	34	7	8	23	24
5	6	15	16	25	26
11	12	21	22	31	32
13	14	29	30	3	4
19	20	35	36	9	10

. (18)

Here is a way of flipping pairs for a general $(n = 2k)$ and $k = 2t + 1$:

Every 2×2 block is of the form $\begin{bmatrix} k & k+1 \\ n+k & n+k+1 \end{bmatrix}$. The permutations of entries in this block can be uniquely represented by three corodinates (a, b, c) where $a = \sum$(column2 -column1) , $b = \sum$(row2 -row1) and $c = \sum$ diagonal $-$ anti diagonal. Thus a, b and c taking possible values from $0, \pm 2, \pm 2n$. The allowed co ordinates are $\mathcal{S} = \{(\pm 2, \pm 2n, 0), (\pm 2n, \pm 2, 0), (\pm 2, 0, \pm 2n), (\pm 2n, 0, \pm 2), (0, \pm 2, \pm 2n), (0, \pm 2n, \pm 2)\}$.

Now the problem of constructing the magic square reduces to the problem of constructing 2n+1 square with entries from $\mathcal{S}$, such that :

- Sum of first co-ordinate along every column is zero.
- Sum of second co-ordinate along every row is zero.
- Sum of third co-ordinate along diagonal and anti diagonal are respectively zero.

One such possibility is :

- Starting with $v = [2; -2; 2; -2; 2; -2; \cdots; 0]_{k\times 1}$, the first co ordinates are $L = [v, v, v, \cdots Cv]_{k\times k}$. Where $C = \begin{bmatrix} 0 & 1 \\ I_{k-1} & 0 \end{bmatrix}$.
- Starting with $w = [0, 2n, -2n, 2n, -2n, \cdots]_{1\times k}$, the second co ordinates are $M = [w; w; w; \cdots wC]_{k\times k}$.
- Starting with $x = [2n; 0; 0; 0; 0; \cdots; 2]_{k\times 1}$, the third co ordinates are $N = [x, 0, 0, 0, \cdots, -Jx]_{k\times k}$. Where J is the left to right flip of identity matrix.

The entries in the blocks of the matrix A_2 permuted according to $P_{k\times k}$ such that $P(i, j) = (L(i, j), M(i, j), N(i, j))$ will give the magic square.

For $n = 6$ we have $k = 3$ and we get the matrix

$$P = \begin{bmatrix} (2, 0, 2n) & (2, 2n, 0) & (0, -2n, -2) \\ (-2, 0, 2n) & (-2, 2n, 0) & (2, -2n, 0) \\ (0, 2n, 2) & (0, -2n, 2) & (-2, 0, -2n) \end{bmatrix}.$$

By permuting the square (19) according to P, we get

33	28	1	2	23	24
27	34	7	8	18	17
12	5	16	15	31	32
6	11	22	21	25	26
14	13	36	35	4	9
19	20	29	30	10	3

. (19)

3 Observations

The duality between the simple square arrangement and magic square also holds in the doubly even magic squares construction which involves reflecting elements about center of the simple square in x patterns (criss cross patterns) [2]. Since there are even number of flips, the matrix $P^2 = I$.

This is because combining two rows (or columns) equidistant from the center in the simple square form corresponding two rows (or columns) in the magic square.

As an example we have order 4 magic square and simple square,

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

,

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

.

With rows in the original simple square being,

$$R_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}, R_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, R_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, R_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Corresponding columns obtained by the transpose of the matrices

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, L_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, L_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, L_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Sum of entries of A along every row R_i and sum along every column L_j is equal to the magic sum 34 (eg : in the green cells). In the magic square rows of A are reshaped as R_i columns reshaped as L_j.

Now we consider the example of order 6 magic square and see that there need not exist such a duality between the simple square and magic square.

For the order six magic square and simple square,

35	1	6	26	19	24
3	32	7	21	23	25
31	9	2	22	27	20
8	28	33	17	10	15
30	5	34	12	14	16
4	36	29	13	18	11

,

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

.

The sum of the entries corresponding to the inverse map of the first row elements is 84, however the magic sum is 111.

4 Conclusion

Using the duality between the simple square arrangement and magic squares odd order magic square is obtained from the lattice of simple square. The duality is also observed for the doubly even magic square. A Conways LUX like method is presented for constructing singly even magic square. However existance of duality for singly even magic square remains an open question. It is also noted by an example that such duality may not exist for all magic squares.

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