Stress Distribution in Gravity Dams

This article describes briefly the attempts that have been made to analyse the stresses in gravity dams so as to take into account the influence of the foundations. The present position in this field is given and possible future developments are indicated

By J. R. RYDZEWSKI, Ph.D., A.M.I.C.E., A.M.ASCE.

THE application of a scientific approach to the design of masonry dams was first apparent in the work of the French engineers, Sazilly, Delocre and Graeff in the middle of the last century. Their dams were designed to resist overturning as well as sliding along the base and were shaped to develop equal maximum vertical compressive stress for both reservoir-full and reservoir-empty conditions. The stresses were calculated on the assumption of a trapezoidal distribution of normal stress on horizontal sections.

Rankine, in 1870, added to this the condition of no tension, resulting in the well-known "middle-third rule" and also pointed out that the maximum compressive stress, on the assumptions used, would act parallel to the face of the dam. Rankine's criteria of design, known as the "classical method," became universally accepted. In France it was elaborated on by Maurice Levy, who proposed that compression at the upstream face should exceed the hydrostatic pressure at the particular section.

The Bouzey dam disaster in France in 1895 aroused an unprecedented interest in the mode of failure of masonry dams. The controversy that raged on that subject in the British technical press and the research activity it stimulated were the subject of Professor Pippard's Unwin Memorial Lecture to the Institution of Civil Engineers[1], to which historians of engineering science should refer.

The failure of the Bouzey dam not only drew the attention of engineers to the problem of uplift, but also subjected the classical method of analysis to severe criticism. Out of that period of intensive research and discussion, which ended in 1910, there emerged two remarkable papers. On the experimental side, Wilson and Gore[2], in 1908, described their experiments with stiff rubber models of a dam together with a portion of its foundations. For the purpose of application of dead load, the profile was divided into a number of imaginary rectangular and triangular sections at the centres of gravity of which vertical loads were applied through pins piercing the model. The hydrostatic load was substituted by a statically equivalent system of concentrated loads, acting through plates bearing on the face of the model. Observations of displacement yielded the required stress distribution, an example of which is given in Fig. 1. It says a good deal for their ingenuity that this method has been used practically unaltered by the United States Bureau of Reclamation some 30 years later.

A solution of the elastic problem of a dam, homogeneous with its foundations, by means of finite difference approximations was attempted by Richardson[3] in 1909. His formulation of the problem left nothing to be desired, but he was not in a position to solve the problem with any accuracy, having considered only a few internal points. This problem was overcome when relaxation techniques, initiated by Southwell, reached an advanced stage of development.

Wolf[4], in 1914, considered the problem of a triangular dam, with a vertical upstream face, resting on perfectly rigid foundations. This he expressed by the condition that the average vertical displacement along the base of the dam should be zero. In order to satisfy this condition he applied secondary forces, in the form of an arbitrary polynomial, to the upstream face, stipulating that their resultant should vanish. In this last condition the secondary forces were regarded as errors to be minimised. One of his results is shown in Fig. 3 (top).

Vogt[5] 1925, and Kalman[6] 1927, demonstrated the incompatibility between the displacements of the base of the dam and those of the foundation surface as calculated by the classical method, suggesting that corrective stresses should be introduced along the base to bring these displacements in conformity with each other.

The case of a triangular dam on an unyielding foundation was also considered by Jakobsen[7] in 1932,

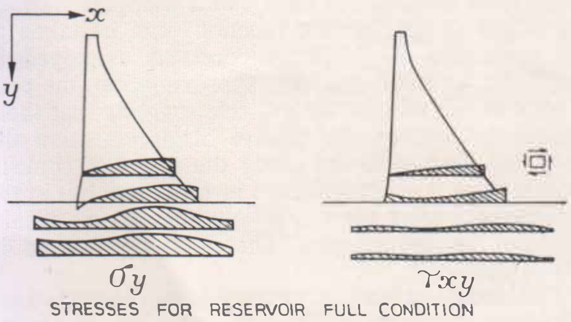

Fig. 1. Stress distribution obtained from displacement observations

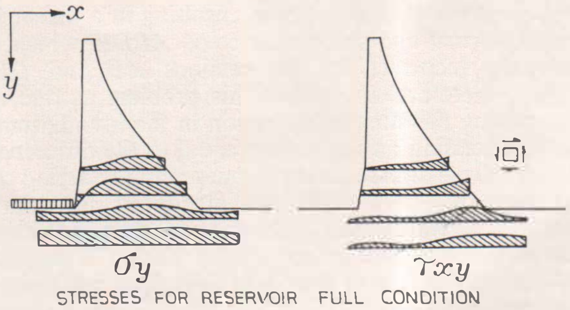

Fig. 2. Stress distribution by slab analogy method

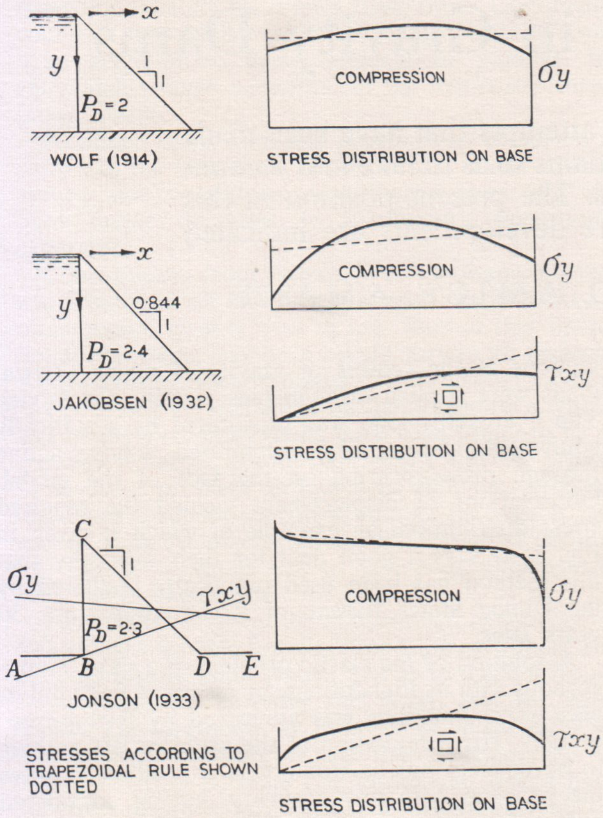

Fig. 3. Stress distribution in a triangular dam

who minimised the strain energy integral using Ritz's method. This approach would reveal the true solution from amongst a family of possible solutions, when it is certain that the true solution is contained in that family. If there is a number of competing families of solutions, the accuracy of the result is as good as the choice of the family. Jakobsen assumed the normal stress on horizontal sections to have a cubic distribution, expressed by a function containing two arbitrary constants. This gave less strain energy than a straight-line distribution, but as it did not take account of strain compatibility conditions, it could not claim to represent the absolute minimum. His solution gave the stress distribution shown in Fig. 3 (centre).

Jonson[8] 1933, gave a brief outline of his attempt to determine the stress distribution in a triangular dam, considered together with a portion of a foundation of the same material. He visualised an elastic area of plane strain on which the profile of the dam and its foundations was inscribed, Fig. 3 (bottom). Stresses conforming with the trapezoidal rule and obeying the boundary conditions on the dam face were applied to the area as a whole, resulting in a distribution of normal and shear stresses on *ABDE*, which is obviously incorrect for the sections *AB* and *DE*. Jonson, therefore, formulated his problem as one of determining the stress distribution in the foundations, due to a loading on the system equal to the difference between these stresses and the ones actually acting on *AB* and *DE*. Like Wolf, he introduced secondary stresses, this time in the form of Airy stress functions, and minimised their effect on the known boundary stresses on the face of the dam. This does not ensure uniqueness of solution, since in considering the region of the foundations, only its upper boundary conditions were specified.

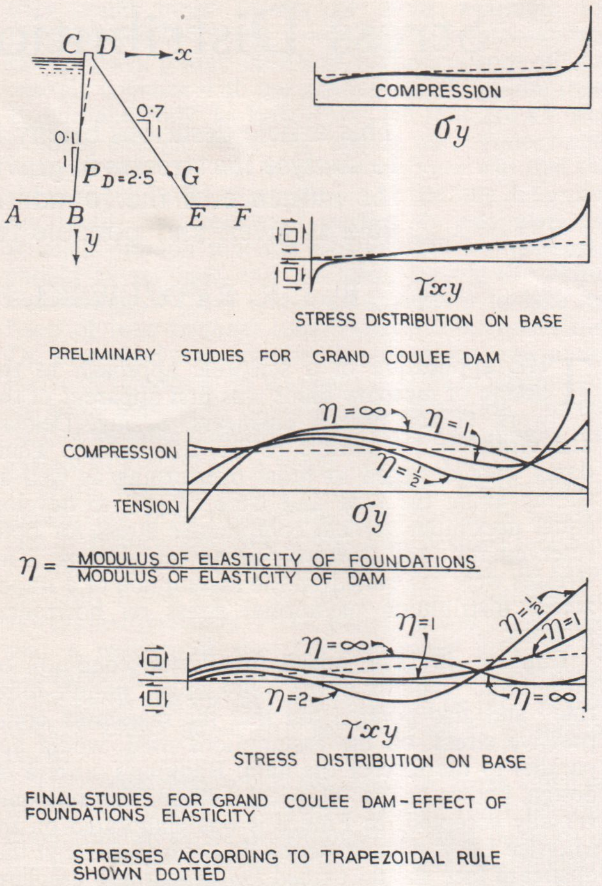

Fig. 4. Stress distribution with corner function

The same criticism can be levelled at the first of two analytical methods proposed by Brahtz[9], in 1936, who introduced correcting stresses in the form of a "corner function" of the general type $\varnothing_c = r^n f_n(\theta)$ for the case of a dam standing on foundations having the same elastic properties. Referring to Fig. 4 (top), an Airy stress "corner function" would be formed in polar co-ordinates, with origin at *B*, in such a way that it did not contribute to the boundary stresses along *AB* and *BC*. Such a stress function contains a double infinity of arbitrary constants which can be used to satisfy further boundary conditions. Brahtz proposed to satisfy the boundary conditions on the downstream face of the dam and the downstream foundation at three points, such as *E, F* and *G*.

Brahtz also mentions briefly in this paper the application of the "corner function" to the more general case when the elastic properties of the foundations differ from those of the dam. For a triangular profile, the origin of the "corner function" can be taken at the apex and the arbitrary constants employed in satisfying, to some specified approximation, the conditions at the third boundary. He suggests that these conditions concern the relative displacement of the dam and its foundation along the plane of contact. Brahtz gives it no prominence in his paper, but in the summing-up after the discussion shows some results obtained in this manner. These are reproduced in Fig. 4 (bottom).

Full details of the last method, however, were published as a Departmental Memorandum for the United States Bureau of Reclamation by Silverman, Zangar, Soerens and Brahtz[10] in 1935. As copies of this are not readily available, the following information may be of interest.

The boundary conditions at the base of the dam were formulated so that, firstly, both the heel and the toe of the dam were in contact with the deformed foundations. Secondly, the magnitude of the vectorial difference between the deflections of the dam and the foundations, when integrated over the base, was made a minimum. To simplify calculations, only four arbitrary constants, out of the infinite number available, were used. Two were made to satisfy the first set of conditions and two the second. As a further simplification, the effect of "corner function" stresses was neglected in the determination of foundation deformations.

Whereas the inaccuracies due to the first simplification can largely be eliminated through repeated application of the process, the second assumption is less easy to justify. It can only be concluded that this was the reason why this investigation received so little attention in Brahtz's later paper.

Tölke[11], in 1938, without making any reference to the work of Brahtz, used an identical form of correcting "corner function" for the problem of a triangular dam, with a vertical upstream face, resting on a semi-infinite foundation of any prescribed elastic properties.

Tölke did not make any simplifying assumptions like Brahtz, and arrived, after a detailed analysis, at equations representing the relative displacements between the dam and its foundation. These relative displacements, in the vertical and horizontal directions, were then minimised by the application of least-squares, i.e. the squares of the relative displacement, when summed over the length of the base, were made a minimum. Four roots of the characteristic equation governing the "corner function" were taken at a time, with the suggestion that further groups of four could be introduced in an iterative process. His results, which were presented in a qualitative manner, are shown in Fig. 5.

It is regrettable that Tölke does not discuss the accuracy of his least-squares solution since, in the author's experience, this is the stage where difficulties are encountered.

In the early 1930's plans were beginning to take shape for the Boulder dam in America. As the structure was to be made on a scale considerably greater than anything built hitherto, methods of stress analysis in accordance with the theory of elasticity were being looked into. In this connection, Westergaard[12], in 1934, drew attention to the analogy between the surface of the Airy stress function and the deflection of an unloaded slab by forces and couples applied along its edges. This stated that assuming similarity, the components of stress in any direction were proportional to the components of curvature at holomorphic points on the analogous slab. It should be noted that for the analogy to hold, body forces have to be replaced by equivalent boundary loads. This can be done, for the harmonic body-force potential of gravity, by means of the analogy between body and boundary forces stated by Biot[13]:

Initially the "slab analogy" by which name the above soon became known, served as an experimental method of stress analysis, details of which were given by the United States Bureau of Reclamation[14] in 1938. Subsequently, experience gained on arch-dam design, with trial-load methods of balancing the load distribution on an orthogonal system of arches and cantilevers to obtain slope and displacement compatibility for the two systems, suggested a similar procedure in conjunction with the slab analogy. Here the boundary loads were applied as usual, but the two sets

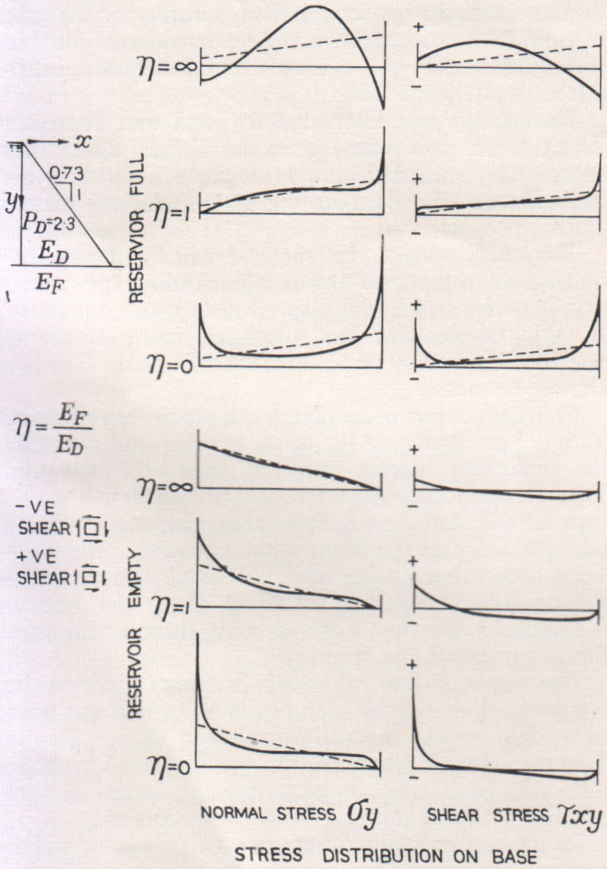

Fig. 5. Stress distribution using least-squares solution

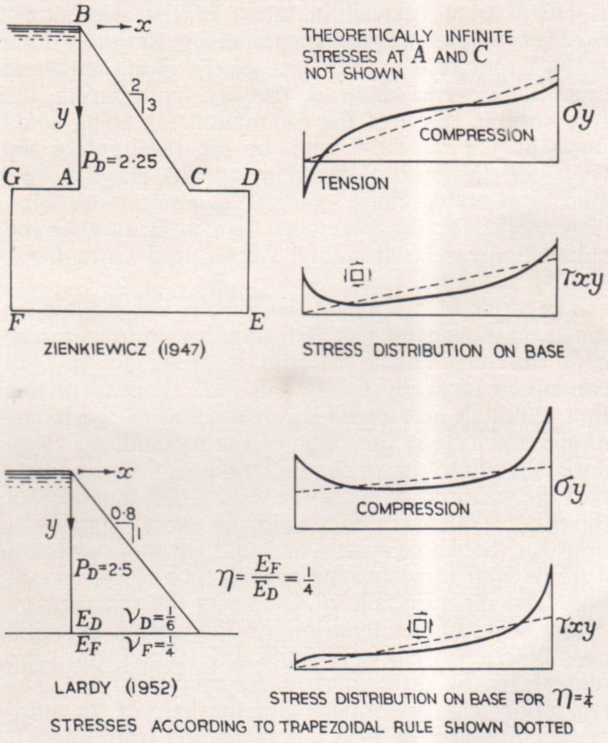

Fig. 6. Stress distribution by relaxation method

of orthogonal beams were brought into conformity with each other by equal and opposite trial loads, thus leaving the slab as a whole unloaded.

To test this method against others then in use, an analysis was made of the gravity dam profile previously studied by Wilson and Gore. The good agreement between the two sets of results, which can be seen by comparing Figs. 1 and 2, encouraged its use on several large projects in the United States. Details of its use on the Boulder dam were given by the United States Bureau of Reclamation[15] in 1939. The success of any trial-load type of analysis depends largely on the availability of suitably trained and experienced staff. The tradition built up in this field in America, mainly in connection with arch-dam analysis, makes it, there, a practical proposition. The method has not found favour in Europe, where the tendency is increasingly towards simplicity in analysis coupled with reliable model tests.

The work of Richardson and the first publications of Southwell on relaxation methods led McHenry[16], in 1944, to investigate successive approximation methods in elasticity from a practical engineer's view.

By replacing a square element of a two-dimensional elastic body by a square, cross-braced, pin-jointed frame, and by assuming 1/3 as a value of Poisson's ratio, he demonstrated that the elastic problem could be solved approximately by solving for displacements of the joints in the analogous lattice. McHenry claimed that the mental picture of clamping and releasing successive internal joints would be of assistance to the designer, but implied that techniques akin to block relaxation could be used. This method could not survive competition with more rigorous relaxation procedures.

In fact, a couple of years later, Zienkiewicz[17] did successfully apply the technique of relaxing the biharmonic equation of elasticity to the case of a dam homogeneous with a finite area of its foundations. Referring to Fig. 6, the boundary conditions along *GABCD* were derived in terms of the Airy stress function in the manner suggested by Brahtz. Along *DEFG*, remote from the base of the dam, the stress conditions were obtained through considering the semi-infinite plane of the foundations as being acted upon by the reservoir load, by the resultant of the forces on the base of the dam and by gravity. This study was not confined to simple geometrical profiles. The results for the Claerwen dam in Britain agreed substantially with those of Wilson and Gore for a similar structure.

The work of a team at the Zurich Polytechnic on the general problem of a dam standing on foundations of a different elastic material is presented, in very condensed form, by Lardy[18] in 1952. It is surprising that although here again the relaxation of the biharmonic equation is the basis of the method, no reference is made to the work of Zienkiewicz. One gathers that the relations between the Airy stress function for the two regions were obtained from energy considerations for the whole system and that various short-cuts were devised to hasten the convergence of the relaxation. The most notable of these was the replacement of the effect of the semi-infinite foundation by influence functions, which, in finite difference form, were represented by three lines of the relaxation net. The relaxation was performed with the help of an automatic computer. In 1958 it was learned from Lardy[19] that details of this analysis were not being published.

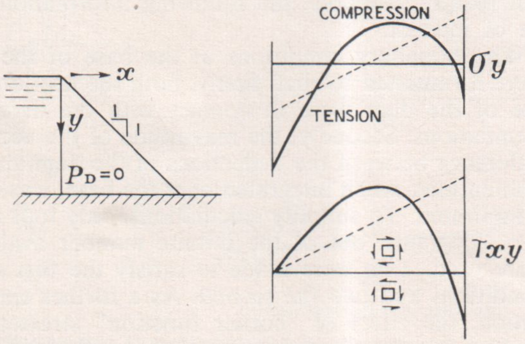

Fig. 7. Stress distribution by energy method

In 1958, Silverman[20], like Jakobsen, approached the problem of a triangular dam on an unyielding foundation through consideration of energy. His method is, however, not restricted by the form of the function for the stress distribution. Like Brahtz and Tölke, he superimposes on the "classical theory" solution a correcting function which produces a self-equilibrating stress system. His correcting function is of the form

$$\varnothing = g(x,\ y)\ f(y),$$

in which $g(x, y)$ is a known function chosen so as to give zero normal and shear stresses at the faces of the dam. The function $f(y)$ is then determined by minimising the strain energy of the dam only, since, of course, the foundations are assumed to undergo no strain. This method is, essentially, that due to Kantorowicz described by Sokolnikoff[21].

Silverman's results for a particular case of hydrostatic loading without body forces are given in Fig. 7. The method appears relatively simple for the case of rigid foundations, but could be considerably less so if the strain energy of an elastic foundation were introduced into the problem.

The author was interested to learn that Professor Zienkiewicz, now working in the United States, has successfully extended his relaxation techniques to the general case of a dam resting on foundations of any elastic properties[22].

The usefulness of this method will be greatly enhanced when the time-absorbing arithmetic of relaxation is performed by high-speed electronic computers. In this connection, the advances being made at Sheffield University under Professor D. N. de G. Allen should be noted.

The author has investigated the possibility of using a digital computer on the same problem but adopting the correcting "corner function" idea of Brahtz and Tölke[23]. Although the computer programme for this type of calculation is simpler than the one involving the relaxation of the biharmonic equation, difficulties were encountered with ill-conditioned equations resulting from boundary matching along the line of foundations. Further work is being done to improve the accuracy of the solution.

The above survey, although it cannot claim to be complete, does give an idea of the effort that has been expended on the theoretical solution of the gravity dam problem. A great deal of ingenuity has also gone into the perfecting of experimental techniques of obtaining the required results from models.

In the author's view there are two ways in which research in this field can assist the designer. Firstly, by providing, with the aid of modern computers, quick methods of analysing a dam profile in accordance with the laws of elasticity, assuming elastic behaviour for the material of the dam. Secondly, by making available to the designer the facilities of model laboratories where experienced staff could carry out tests on models of the selected design, but made from a material closely resembling the material of the prototype.

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Indus. Wapda Public Relations, The Pipals, Lahore, Pakistan. Annual subscription Rs. 5.

In our May issue we commented on the first Annual Report of the West Pakistan Water and Power Development Authority (Wapda). We have now received the first issue of a house journal issued by Wapda Public Relations, which has been established to deal more or less informally with the aims and activities of the Authority. In this first issue the task facing the Authority is admirably surveyed in an article "A New Approach" by Ghulam Faruque, Wapda's Chairman, and other articles are concerned with rural electrification, land reclamation and matters of general interest.

"Seven Over the Severn" is the title of a recent publication issued by the Aluminium Wire and Cable Co. Ltd., of 30, Charles II Street, St. James's Square, London, S.W.1. The publication, which is amply illustrated, gives an account of the construction of the transmission lines across the Severn. Seven unjointed lengths of conductor, each $2\frac{1}{4}$ miles long and weighing 23 tons, were manufactured in the company's Swansea factory, and form the important link of the 275 kV supergrid system across the rivers Severn and Wye. The crossing of these two rivers at their point of confluence was a task of considerable magnitude, involving three long spans, one of which exceeds a mile in length—the longest in Britain. The project demanded conductors of special characteristics. They are the largest both in cross-section and in length yet erected in Britain and their manufacture involved a number of unique problems. Each conductor consists of 78 aluminium wires of 0·113 in. diameter arranged in two layers around a core of 91 high-tensile galvanised steel wires also of 0·113 in. diameter. The overall conductor has a diameter of 1·695 in., weighs 4·30 lb. per ft. with grease, and has a breaking load of 227,000 lb.

A second publication, No. G25, called "What AWCO Makes," outlines the variety of products manufactured and sets out the present extent of their production.

Technoexport of Praha, Czechoslovakia, have issued a well-produced brochure dealing with the various water-power plants they have installed in different parts of the world. The brochure is illustrated by coloured photographs and describes complete power plants, automatic governing gear, weir equipment, control valves, segment crest gates, exciters, etc., and transformers.

The Ironfoundry Handbook. The Standard Catalogue Co. Ltd., 26 Bloomsbury Way, London, W.C.1, have compiled and published a new reference book for the ironfoundry trade. It contains technical information on the industry and includes a guide for the facilities, capabilities and capacities of foundries in the United Kingdom and a buyers guide for all types of foundry equipment and materials. This information, which is comprehensive, can be of assistance to designers in the field. The book contains 390 pp. and costs 42s. for a single copy, or 35s. each for two or more copies. It will be published once every two years.