THEORY OF TURBINES

WOOD

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Remark.

About forty-five years ago M. Poncelet made a solution of the Fourneyron turbine which, for its thoroughness and the directness of its analysis, has become classical (*Comptes Rendus*, 1838). But that writer neglected the frictional (and other) resistances within the wheel, and assumed that the buckets, or passages in the wheel, were constantly full. The former is an important element in the theory, and its consideration makes the analysis but little more complicated.

Weisbach, in his *Hydraulic Motors*, gives a solution in which frictional resistances are involved, and the sections of the stream at the outlet of the supply chamber, the entrance into the wheel, and all the sections of the buckets are determined when the wheel runs for best efficiency. The formulas, however, are so complex that but little practical knowledge can be gained from their general discussion. I have, therefore, assumed that the wheels here discussed have about the proportions made for commercial purposes, and deduced certain numerical results which are entered in tables; and a simple examination of these furnishes certain desirable information.

The driving power here considered is that of an incompressible fluid, which in practice will be water. The steam turbine or those driven by a compressible fluid are not in practice constructed like water turbines, and no theory for such is here attempted.

HYDRAULIC MOTORS.

1. The motors here analyzed may be called "reaction wheels," or "pressure turbines." Some writers call those wheels "pressure turbines" in which the water has a "free surface," and these by others are called "turbines of free deviation. (Weisbach, *Hydraulics, etc.*, p. 421.) The former term is somewhat ambiguous when applied to wheels in which the water in the buckets has a free surface, and, therefore, the term "free deviation" will be applied to such.

There will first be given a *general solution* of the "pressure turbine," and the other turbines will be considered as special cases of the more general one.

2. *Notation.*

Let Q be the volume of water passing through the wheel per second,

δ, the weight of unity of volume of the water, or $62\frac{1}{2}$ pounds per cubic foot; then

$\delta Q = W$ will be the weight of water passing through the wheel per second,

h_1 be the head in the supply chamber above the entrance to the buckets,

h_2, the head in the tail race above the exit from the bucket,

z_1, the fall in passing through the buckets,

$H = h_1 + z_1 - h_2$, the effective head,

U, the useful work done by the water upon the wheel,

R, the work lost by frictional resistances, whirls, etc.,

μ_1, the coefficient of resistance along the guides,

μ_2, the coefficient of resistance along the buckets,

r_1, the radius of the initial rim,

r_2, the radius of the terminal rim,

ρ, the radius to any point of the bucket,

$n = r_1 \div r_2$, the ratio of the initial to that of the terminal radius,

V, the velocity of the water issuing from supply chamber,

v_1, the initial velocity of the water in the bucket in reference to the bucket,

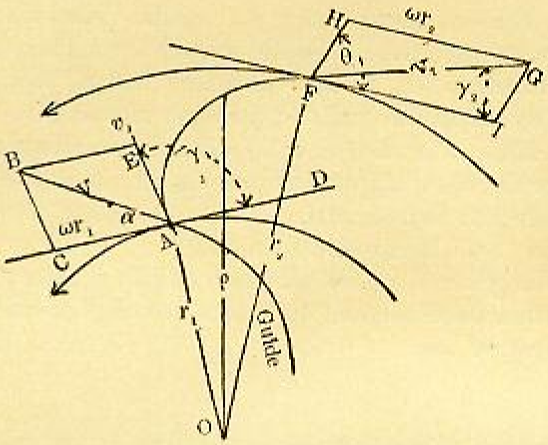

Fig. 1.

v, the velocity along the bucket at any point,

v_2, the terminal velocity in the bucket,

w, the velocity of exit in reference to the earth,

ω, the angular velocity of the wheel,

α, terminal angle between the guide and initial rim $= CAB$,

γ_1, angle between the initial element of bucket and initial rim $= EAD$,

$\gamma_2 = GFI$, the angle between the terminal rim and terminal element of the bucket,

$\theta = HFI$, angle between the terminal rim and actual direction of the water at exit,

p_1, the pressure of water at entrance of the bucket per unit area,

p, the pressure of water at any point of the bucket,

p_2, the pressure of water at exit,

p_a, the pressure of an atmosphere,

$a = cb$, the arc subtending one gate opening, Fig. 3,

Measurement of Flow.

$dQ = \sqrt{2gy}\, dx\, dy\, \mu$

$Q = \int \sqrt{2g} \int_{h_2}^{h_1} \int_0^b y^{\frac{1}{2}}\, dy\, dx\, \mu$

$Q = \frac{16}{3} b \left(h_1^{\frac{3}{2}} - h_2^{\frac{3}{2}}\right) \mu$

$Q = .4 \times 8 \times 6 \left(2^{\frac{3}{2}} - \frac{1}{2}^{\frac{3}{2}}\right) = 47.5584$ cu ft per sec

Fall = 16' reqd. H.P.

$HP = \frac{47.6 \times 62.4 \times 16}{550} = 86.34$

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a_1, the arc subtending one bucket at entrance. (In Fig. 3, a and a_1 appear to be the same but in practice they are usually different, a being greater than a_1.)

$a_2 = gh$, the arc subtending one bucket at exit,

$K = bf$, normal section of passage, it being assumed that the passages and buckets are *very* narrow,

$k_1 = bd$, initial normal section of bucket,

$k_2 = gi$, terminal normal section,

Y, the depth of K, y_1 of k_1, and y_2 of k.

Then

$$\left.\begin{aligned} K &= Ya \sin \alpha ; \\ k_1 &= y_1 a_1 \sin \gamma_1 ; \\ k_2 &= y_2 a_2 \sin {}_2 , \end{aligned}\right\} \quad (1)$$

ωr_1 = velocity of initial rim,

ωr_2 = velocity of terminal rim.

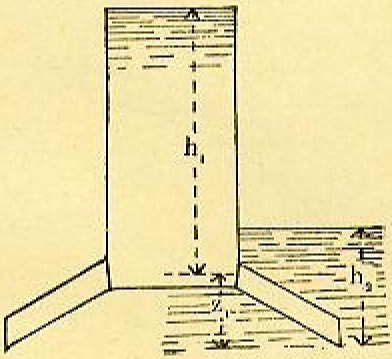

FIG. 2.

3. *General Solution.*

Beginning with the pressure on the top of the supply chamber, the relation between the heads, actual and virtual, will be determined to the point of discharge from the wheel.

The pressure per unit on the upper surface of the supply chamber will be that of the atmosphere, or

$$p_a ,$$

and the corresponding virtual head in terms of a column of water will be

$$\frac{p_a}{\delta} .$$

The head in the supply chamber above the entrance to the wheel will be

$$h_1 ;$$

therefore, the total head above the initial element of the bucket will be

$$h_1 + \frac{p_a}{\delta}.$$

This head produces an actual pressure p_1 at the entrance to the bucket and the velocity V of exit from the guides; hence, according to Bernoulli's theorem, the heads due to the pressure p_1 and velocity V, will equal the former, or

$$h_1 + \frac{p_a}{\delta} = \frac{p_1}{\delta} + \frac{V^2}{2g}; \quad \ldots \ldots \quad (2)$$

$$\therefore p_1 = p_a + \delta h_1 - \delta \frac{V^2}{2g}, \quad \ldots \ldots \quad (3)$$

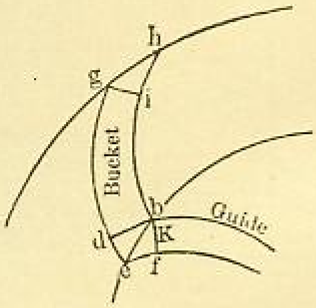

Fig. 3.

which will be the theoretical pressure at entrance to the bucket if friction be neglected. Represent the head lost by friction by

$$\mu_1 \frac{V^2}{2g},$$

which must also be overcome by the head in the supply chamber, so that we have, by adding it to the second member of (2) and transforming,

$$(1 + \mu_1) V^2 = 2gh_1 + 2g\left(\frac{p_a - p_1}{\delta}\right). \quad \ldots \quad (4)$$

The triangle of velocities ABC, Fig. 1, gives

$$V = \frac{\sin \gamma_1}{\sin(\alpha + \gamma_1)} \omega r_1 = \frac{\sin \gamma_1}{\sin(\alpha + \gamma_1)} n\omega r_2 \quad \ldots \quad (5)$$

$$v_1 = \frac{\sin \alpha}{\sin(\alpha + \gamma_1)} \omega r_1. \quad \ldots \ldots \quad (6)$$

The relation between the initial and terminal velocities in the bucket involves the velocity of the wheel and the pressure in the bucket.

Let m be an elementary mass at a distance ρ from the axis of the wheel, then will the centrifugal force be

$$m\omega^2\rho,$$

and if this element by moving a distance ds in the tube also moves outward a distance $d\rho$, the work done by the centrifugal force will be

$$m\omega^2\rho d\rho.$$

If the tube (or bucket) be inclined downward, the work done (or energy acquired) by the weight in falling a distance dz will be

$$mgdz.$$

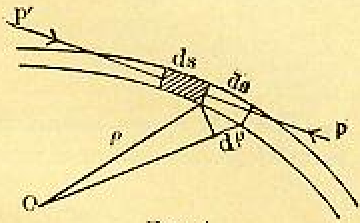

FIG. 4.

These two works will be expended in the following ways:

a. Increasing the energy of the water in the tube in reference to the tube by an amount

$$\tfrac{1}{2}\, md\,(v^2).$$

b. In doing work against the difference of pressures on the two faces of the element, and considering the back pressure p greater than the forward pressure p', the work will be

$$mg\frac{dp}{\delta},$$

where $dp \div \delta$ is equivalent to a head through which mg would work.

c. In overcoming frictional resistance. The law of frictional resistances is not well known, but is assumed to vary as the energy of the mass and wetted perimeter. The perimeter is here discarded, hence the work will be

$$\mu_2 \cdot \tfrac{1}{2}\, mv^2 ds.$$

Hence we have

$$mgdz + m\omega^2 \rho d\rho = \tfrac{1}{2}\, md\,(v^2) + mg\,\frac{dp}{\delta} + \tfrac{1}{2}\,\mu_2 mv^2 ds. \quad . \quad . \quad . \quad (7)$$

But the last term cannot be integrated unless v be a known function of s, and since this is not known, we make $v = v_2$, the terminal velocity. The coefficient μ_2 is determined independently of the length, and includes the value $\mu_2 s$, when s is the length of a bucket.

Integrating between initial and terminal limits gives

$$z_1 + \frac{\omega^2(r_2^2 - r_1^2)}{2g} = \frac{v_2^2 - v_1^2}{2g} + \frac{p_2 - p_1}{\delta} + \mu_2\,\frac{v_2^2}{2g}. \quad . \quad (8)$$

The fall z_1 is so small in practice compared with the next term of the equation, that it may, and will, be omitted, giving

$$(1 + \mu_2)\, v_2^2 = v_1^2 + \omega^2\,(r_2^2 - r_1^2) - 2g\,\frac{p_2 - p_1}{\delta}, \quad . \quad . \quad . \quad (9)$$

which gives v_2.

At exit the pressure will be

$$p_2 = p_a + \delta h_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The velocity of exit, relative to the earth, will be

$$w^2 = v_2^2 + \omega^2 r_2^2 - 2v_2 . \omega r_2 \cos \gamma_2. \quad . \quad . \quad . \quad . \quad (11)$$

The work done upon the wheel will be the initial (potential) energy of the water less the energy in the water as it quits the wheel, still further diminished by the energy due to frictional losses; or

$$U = \delta QH - \delta Q\,\frac{w^2}{2g} - R. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$R = \mu_1 Q\,\frac{V^2}{2g} + \mu_2 \delta Q\,\frac{v_2^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

A sufficient number of equations have now been established to find, by elimination, the useful work U in terms of the angular velocity ω and known constants; and the efficiency will be U divided by the *theoretical* work the water was capable of doing. Performing the operations, there will be found

$$E = \frac{U}{\delta Q H} =$$

$$\frac{r_2\omega}{gH\sqrt{1+\mu_2}} \times \left\{ \frac{\sqrt{1+\mu_2}\left\{-1 + \frac{\cos\alpha \sin\gamma_1}{\sin(\alpha+\gamma_1)}\left(\frac{r_1}{r_2}\right)^2\right\} r_2\omega + \cos\gamma_2 \times}{\sqrt{2gH + \left\{1 - \left(\frac{2\cos\alpha\sin\gamma_1}{\sin(\alpha+\gamma_1)} + \mu_1 \frac{\sin^2\gamma_1}{\sin^2(\alpha+\gamma_1)}\right)\left(\frac{r_1}{r_2}\right)^2\right\} r^2{}_2\omega^2}} \right\} . \quad (14\ *$$

* To deduce (14).

$$\frac{2gU}{\delta Q} = 2gH - w^2 - \mu_1 V^2 - \mu_2 v_2^2, \text{ from (13 and 12)} \quad = 2gH - (1+\mu_2)v^2_2 - \omega^2 r^2_2 + 2\omega r_2 \cos\gamma_2 . v_2 - \mu_1 V^2 \text{ from (11)}$$

$$= 2gH - v_1^2 - 2\omega^2 r^2_2 + \omega^2 r_1^2 + 2g\frac{p_a - p_1}{\delta} + 2gh_2 . - \mu_1 V^2 + \frac{2\omega r_2 \cos\gamma_2}{\sqrt{1+\mu_2}}\sqrt{v_1^2 + \omega^2(r^2_2 - r_1^2) - 2g\frac{p_a - p_1}{\delta} - 2gh_2}$$

(by means of 9 and 10)

From (4).

$$2g\frac{p_a - p_1}{\delta} = (1+\mu_1)V^2 - 2gh_1.$$

Substituting this, and v_1 from equation (6) and V from (5) gives

$$\frac{2gU}{\delta Q} = -2\omega^2 r^2_2 + \omega^2 r_1^2 - \left(\frac{\sin^2\alpha}{\sin^2(\alpha+\gamma_1)} - \frac{\sin^2\gamma_1}{\sin^2(\alpha+\gamma_1)}\right)\omega^2 r_1^2$$

$$+ \frac{2\omega r_2\cos\gamma_2}{\sqrt{1+\mu_2}}\sqrt{\omega^2(r_2^2 - r_1^2) + 2gH + \left(\frac{\sin^2\alpha}{\sin^2(\alpha+\gamma_1)} - \frac{\sin^2\gamma_1}{\sin^2(\alpha+\gamma_1)}\right)\omega^2 r_1^2 - \mu_1\frac{\sin^2\gamma_1}{\sin^2(\alpha+\gamma_1)}\omega^2 r_1^2}.$$

$$E = L\left[-M^2 r_2\omega + \cos\gamma_2\sqrt{2gH + N^2 r_2^2\omega^2}\right] r_2\omega, \quad . \quad . \quad . \quad (15)$$

when

$$\left.\begin{aligned} L &= \frac{1}{gH\sqrt{1+\mu_2}}, \\ -M^2 &= \sqrt{1+\mu_2}\left[-1 + \frac{\cos\alpha\sin\gamma_1}{\sin(\alpha+\gamma_1)}\left(\frac{r_1}{r_2}\right)^2\right], \\ N^2 &= 1 - \left(\frac{2\cos\alpha\sin\gamma_1}{\sin(\alpha+\gamma_1)} + \mu_1\frac{\sin^2\gamma_1}{\sin^2(\alpha+\gamma_1)}\right)\left(\frac{r_1}{r_2}\right)^2 \end{aligned}\right\} \quad (15a)$$

For maximum efficiency make $dE \div d\omega = 0$ in (15) and solve for ω, calling this particular value ω', then

$$\omega' = \frac{\sqrt{gH}}{r_2}\sqrt{\frac{M^2 - \sqrt{M^4 - N^2\cos^2\gamma_2}}{N^2\sqrt{M^4 - N^2\cos^2\gamma_2}}}, \quad . \quad . \quad . \quad (16)$$

which value substituted in equation (15) will give the maximum efficiency. Then equations (5), (6), become

$$V = \frac{\sin\gamma_1}{\sin(\alpha+\gamma_1)}\omega' r_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$v_1 = \frac{\sin\alpha}{\sin(\alpha+\gamma_1)}\omega' r_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Also from (9), (10), (4), (17), (18),

But

$$\frac{\sin^2\alpha - \sin^2\gamma_1}{\sin^2(\alpha+\gamma_1)} = \frac{\sin^2\alpha - \sin^2\gamma_1}{\sin^2\alpha - \sin^2\gamma_1 + 2\sin(\alpha+\gamma_1)\cos\alpha\sin\gamma_1}$$

$$= 1 - \frac{2\sin(\alpha+\gamma_1)\cos\alpha\sin\gamma_1}{\sin(\alpha+\gamma_1)\sin(\alpha-\gamma_1) + 2\sin(\alpha+\gamma_1)\cos\alpha\sin\gamma_1}$$

$$= 1 - \frac{2\cos\alpha\sin\gamma_1}{\sin(\alpha+\gamma_1)},$$

which substituted above will give equation (14).

$$\max E = \frac{1}{\sqrt{1+\mu^2}\; N^2}\left\{M^2 - \left[M^4 - N^2 \gamma^2\right]^{1/2}\right\}$$

$$L = \frac{1}{32 \times 16\sqrt{1.2}} = .0018 \text{ nearly}$$

$$M^2 = 1.095\left[-1 + \frac{\cos 10° \; 190°}{\sin 100°}\left(\frac{3}{2}\right)^2\right] = 1.369$$

$$N^2 = 1 - \left(2 + .1\frac{1}{.9848^2}\right)\frac{9}{4} = 1 - 4.73 = -3.73$$

$$\omega' = \frac{16\sqrt{2}}{2}\sqrt{\frac{-1.369 - \sqrt{1.369^2 + 3.737^2 \; 12°}}{-3.73\sqrt{1.369^2 + 3.737^2 \; 12°}}}$$

$$\omega' = \frac{16 \times 1.414 \times .4255}{2} = 7.3788 \text{ per sec.}$$

$$V = \frac{1\, y_1}{1(x+y_1)}\,\omega' r_1 = \frac{1}{.9848}\, 7.3788 \times 3$$

$$V = 22.48' \text{ per sec.}$$

$$V_2 = .1763 \times 7.3788 \times 3 = 3.36$$

$$V_1 = \sqrt{\frac{1}{1.2}} \times \sqrt{1024 + (4 - 2 \times 9 - .10301 \times 9)\,7.3788^2}$$

$$V_1 = .9128 \times 14.49 = 13.23' \text{ per sec}$$

$$E_{max} = \frac{1}{1.095 \times -3.73}\left(-1.369 - \sqrt{1.874 + 3.737^2\; 12°}\right)$$

$$E_m = \frac{3.702}{1.095 \times -3.73} = 90.6 + \%$$

$$k_1 = \frac{2}{13,23} = ,151 \square'$$

$$k_2 = \frac{2}{3,36} = ,6 \square'$$

$$y_1 = \frac{,6 \times 144}{10\frac{3}{16} \times 1} = 8,4''$$

$$y_2 = \frac{,151 \times 144}{5 \times ,2} = 21,7''$$

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$$v_2 = \sqrt{\frac{1}{1+\mu_2}}$$

$$\times \sqrt{2gH + \left(r_2^2 - 2\frac{\cos\alpha \sin\gamma_1}{\sin(\alpha+\gamma_1)} r_1^2 - \mu_1 \frac{\sin^2\gamma_1}{\sin^2(\alpha+\gamma_1)} r_1^2\right)\omega^2} \quad . \quad (19)$$

The normal sections of the buckets will be

$$K = \frac{Q}{V}; \quad k_1 = \frac{Q}{v_1}; \quad k_2 = \frac{Q}{v_2}; \quad k = \frac{Q}{v} \quad . \quad . \quad . \quad (20)$$

The depths of those sections will be

$$Y = \frac{K}{A \sin\alpha}, \quad y_1 = \frac{k_1}{a_1 \sin\gamma_1}, \quad y_2 = \frac{k_2}{a_2 \sin\gamma_2}, \quad . \quad (21)$$

DISCUSSION.

4. *Three simple systems are recognized.*

$r_1 < r_2$, called *outward flow.*
$r_1 > r_2$, called *inward flow.*
$r_1 = r_2$, called *parallel flow.*

The first and second may be combined with the third, making a *mixed* system. The third, in theory, is really an inward or outward flow, with an indefinitely narrow crown, although the analysis applies to a parallel flow wheel, in which the width is indefinitely small, and depth small compared with the total head.

5. *Value of γ_2, the quitting angle.*

Equations (14) and (15) show that the efficiency is increased as $\cos\gamma_2$ is increased, or as γ_2 decreases, and is greatest for $\gamma_2 = 0$. Hence, theoretically, the terminal element of the bucket should be tangent to the quitting rim for best efficiency. This, however, for the discharge of a finite quantity of water, would require an infinite depth of bucket, as shown by the third of equations (21). In practice, therefore, this angle must have a finite value The larger the diameter of the terminal rim the smaller may be this angle for a given depth of wheel and given quantity of water discharged. Theoretical considerations then would require, for best efficiency, a very large diameter for the quitting rim, and a very small angle, γ_2, between the terminal element of the bucket and the rim; but commercial considerations require some sacrifice of best efficiency to cost, so that a smaller diameter and larger angle of discharge is made. If wheels are of the same diameter and depth, the inward flow wheel requires a larger quitting angle for the same volume of water than the outward flow, since the discharge rim will be smaller in the former than in the latter wheel, and the velocity v_2, eq. (19), will also be less. In practice γ_2 is from 10° to 20°.

6. *Relation between γ_2 and ω'.*

Equation (16) when put under the form

$$\omega' = \frac{\sqrt{gH}}{Nr_2}\sqrt{\frac{1}{\sqrt{1 - \frac{N^2 \cos^2 \gamma_2}{M^2}}} - 1} \quad . \quad . \quad . \quad (22)$$

shows that ω' increases as γ_2 decreases, and is largest for $\gamma_2 = 0$; that is, in a wheel in which all the elements except γ_2 are fixed, *the velocity of the wheel for best effect must increase as the quitting angle of the bucket decreases.*

If the terminal element be radial, then $\gamma_2 = 90°$, and equation (22) gives $\omega' = 0$; that is, for minimum efficiency the wheel must be at rest, and no work will be done.

7. *Values of $\alpha + \gamma_1$.*

If $\alpha + \gamma_1 = 180°$, and α and γ_1 both finite, then will M and N in (15a) both be infinite; but equation (5) gives

$$\omega = \frac{\sin 180°}{\sin \gamma_1} \cdot \frac{V}{r_1} = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

that is, the wheel will have no motion, and no work will be done. If $\alpha + \gamma_1 = 180°$, then the terminal element of the guide and the initial element of the bucket have a common tangent, in which case the stream can flow smoothly from the former into the latter only when the wheel is at rest. (See Fig. 5.)

If $\alpha + \gamma_1$ exceed 180°, ω' would be negative, and it would be necessary to rotate the wheel backwards in order that the water should flow smoothly from the guide into the bucket.

It follows, then, that $\alpha + \gamma_1$ must be less than 180°, but the best relation cannot be determined by analysis; however, since the water should be deflected from its course as much as possible from its entering to its leaving the wheel, the angle α for this reason should be as small as practicable.

8. *Values of α.*

If $\alpha = 0$, equation (14) will reduce to

$$E = \frac{\omega}{gH\sqrt{1+\mu_2}}\Big[\sqrt{1+\mu_2}\,(-r_2^2 + r_1^2)\,\omega$$

$$+ r_2 \cos\gamma_2 \sqrt{2gH + (r_2^2 - 2r_1^2 - \mu_1 r_1^2)\,\omega^2}\Big] \quad . \quad . \quad (24)$$

which is independent of γ_1; hence, for this limiting case, the efficiency will be independent of the initial angle of the bucket. This is because the water enters the wheel tangentially and therefore has no radial component that would give an initial velocity in the bucket; and equation (18) shows that the initial velocity v_1 would be zero, while (17) shows that the velocity of the initial rim must equal that of the water flowing from the guides, or

$$V = \omega' r_1.$$

For the limiting, or critical case,

$$\alpha = 0,\ \gamma_2 = 0,\ \mu_1 = 0,\ \mu_2 = 0,$$

the velocity producing maximum efficiency will be, from equation (16),

$$r_1\omega' = \sqrt{gH}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

or the velocity of the initial rim, if the wheel be frictionless, will be that due to half the head in the supply chamber.

If $r_2^2 = 2r_1^2$, then

$$r_2\omega' = \sqrt{2gH}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

or the velocity of the terminal rim will equal that due to the head. Substituting in (19) the values $\alpha = 0$, $\gamma_2 = 0$, $\mu_1 = 0$, $\mu_2 = 0$, $r_2^2 = 2r_1^2$, and it will reduce to

$$v_2 = \sqrt{2gH}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

as it should.

The following table gives the values of quantities for the three classes of wheels:

TABLE I.

$\alpha = 0$, $\gamma_2 = 0$, $\mu_1 = 0$, $\mu_2 = 0$.

Dimensions of Wheel.	Velocity of		Velocity of Exit from Guide V.	Velocity in Bucket.		Velocity of Exit. w.	Efficiency. E.
	Inner Rim.	Outer Rim.		Initial v_1.	Terminal v_2.		
	$\omega' r_1$	$\omega' r_2$					
$r_1 = \sqrt{\frac{1}{2}} r_2$	$\sqrt{gH}$	$\sqrt{2gH}$	$\sqrt{gH}$	0.00	$\sqrt{2gH}$	0.00	1.000
$r_1 = r_2$	$\sqrt{gH}$	$\sqrt{gH}$	$\sqrt{gH}$	0.00	$\sqrt{gH}$	0.00	1.000
	$\omega' r_2$	$\omega' r_1$					
$r_1 = 1.4 r_2$	$0.714\sqrt{gH}$	$\sqrt{gH}$	$\sqrt{gH}$	0.00	$714\sqrt{gH}$	0.00	1.000

In the first case the inner rim is the initial one, in the third case the outer rim is initial, it being an inward flow wheel.

Since, in this case, the velocity of admission to the wheel in reference to the earth is that due to half the head in the supply chamber, and the velocity of exit is zero, it follows that the energy due to the velocity is all imparted to the wheel; and the energy due to the remaining half of the head is imparted to the wheel by pressure in the wheel. If the velocity of entrance to the wheel be that due to the head, or $V^2 = 2gH$ then will no energy be imparted to the wheel on account of pressure exerted by any part of the head H, but if $V^2 < 2gH$, then will some of the work be done by this pressure, w being zero. For the cases in Table I., *the energy imparted to the wheel will be due one-half to velocity and one-half to pressure*; or in symbols,

$$U = \tfrac{1}{2}MV^2 + W.\tfrac{1}{2}H$$

$$= \tfrac{1}{2}\frac{W}{g}.gH + \tfrac{1}{2}WH = WH, \quad . \quad . \quad (28)$$

or, the entire potential energy of the water will be expended in work upon the wheel.

Whenever $V^2 < 2gH$, the pressure at entrance must exceed the external pressure at exit, and

$$\frac{H - \frac{V^2}{2g}}{H}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

then will be the part of the head producing pressure in the wheel.

In practice, α cannot be zero and is made from 20° to 30°. When other elements of the wheel are fixed, the value of α may be determined so as to secure a certain amount of initial pressure in the wheel, as will be shown hereafter.

The value $r_1 = 1.4r_2$ makes the width of the crown for internal flow about the same as for $r_1 = \sqrt{\tfrac{1}{2}}r_2$ for outward flow, being approximately 0.3 of the external radius.

9. *Values of* μ_1 *and* μ_2.

The frictional resistances depend not only upon the construction of the wheel as to smoothness of the surfaces, sharpness of the angles, regularity of the curved parts, but also upon the manner it is run; for if run too fast, the initial elements of the wheel will cut across the stream of water, producing eddies and preventing the buckets from being filled, and if run too slow, eddies and whirls may be produced and thus the effective sections be reduced. These values cannot be definitely assigned beforehand, but Weisbach gives for good conditions,

$$\mu_1 = \mu_2 = 0.05 \text{ to } 0.10. \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

They are not necessarily equal, and μ_1 may be from 0.05 to 0.075, and μ_2 from 0.06 to 0.10, or values near these.

10. *Values of* γ_1.

It has already been shown that γ_1 must be less than $180° - \alpha$. If $\gamma_1 = 90°$, equation (14) shows that the efficiency of the frictionless wheel will be independent of α. The effect of different values for γ_1 is best observed from numerical results as shown in the following table:

TABLE II.

Let $\alpha = 25°$, $\gamma_2 = 12°$, $\mu_1 = \mu_2 = 0.10$.

Initial Angle. γ_1. (1)	$r_1 = r_2\sqrt{\frac{1}{2}}$.			$r_1 = 1.4r_2$.		
	$\omega' r_2$. (2)	E. (3)	$\omega' r_1$. (4)	$\omega' r_2$. (5)	E. (6)	$\omega' r_1$. (7)
60°	$1.322\sqrt{gH}$	.812	$.934\sqrt{gH}$	$.780\sqrt{gH}$	.911	$1.092\sqrt{gH}$
90°	1.226 "	.827	.866 "	.689 "	.908	.964 "
120°	1.078 "	.838	.762 "	.576 "	.898	.806 "
150°	.518 "	.744	.366 "	.271 "	.752	.379 "

The values $\omega' r_2$ in columns (2) and (5) are velocities for the terminal rim, which in column (2) are for the exterior rim, but for column (5) it is the interior rim, while column (7) is for the exterior rim.

Columns (2) and (7) show that the velocity of the outer rim is less, for maximum effect, for the inflow than for the outflow, for the same size wheel.

Column (3) shows that the efficiency, E, decreases as the initial angle of the bucket, γ_1, increases up to 120°. This maximum will be for this wheel with this amount of friction.

Column (6) shows that for the inflow wheel the efficiency continually decreases as γ_1 increases. If the head and quantity of water discharged be constant, the work would be proportional to the efficiency; for, from equation (14),

$$U = \delta QHE \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

The effect of γ_1 on the velocities is shown in Table III.

TABLE III.

Let $\alpha = 25°$, $\gamma_2 = 12°$, $\mu_1 = \mu_2 = 0.10$, $Q = 1$.

Initial Angle. γ_1	$r_1 = r_2\sqrt{\frac{1}{2}}$.						$r_1 = 1.4\,r_2$.					
	$\frac{V}{\sqrt{gH}}$	$\frac{v_1}{\sqrt{gH}}$	$\frac{v_2}{\sqrt{gH}}$	$K \times \sqrt{gH}$	$k_1 \times \sqrt{gH}$	$k_2 \times \sqrt{gH}$	$\frac{V}{\sqrt{gH}}$	$\frac{v_1}{\sqrt{gH}}$	$\frac{v_2}{\sqrt{gH}}$	$K \times \sqrt{gH}$	$k_1 \times \sqrt{gH}$	$k_2 \times \sqrt{gH}$
60°	.820	.396	1.447	1.219	2.525	.691	.959	.463	.761	1.043	2.100	1.314
90°	.955	.403	1.378	1.047	2.481	.725	1.063	.449	.676	.940	2.227	1.479
120°	1.150	.560	1.153	.869	1.785	.874	1.217	.593	.605	.821	1.686	1.653
150°	2.100	1.775	.621	.476	.563	1.610	2.060	1.741	.296	.485	.574	3.378

For commercial considerations it may be necessary to sacrifice some efficiency to save on first cost, and to avoid making the wheel unwieldy.

From equation (4) it appears that the pressure in the wheel at entrance, p_1, diminishes as the velocity of admission, V, increases, and, according to equation (5), V depends upon γ_1 when α is fixed. Since the crowns are not fitted air tight nor water tight it is desirable that p_1 should exceed the pressure of the atmosphere when the wheel runs in free air, or the pressure $p_2 + p_a$ when submerged, to prevent air or water from flowing in at the edge of the crown. It will be shown hereafter, in discussing the pressures in the wheel, that we should have

$$-\tan \gamma_1 > \tan 2\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

or, $180^\circ - \gamma_1 > 2\alpha,$

or, $\gamma_1 < 180^\circ - 2\alpha.$

If $\alpha = 30,$

then $\gamma_1 < 120^\circ.$

To be on the safe side, the angle γ_1 may be 20 or 30 degrees less than this limit, giving

$$\gamma_1 = 180^\circ - 2\alpha - 25 \text{ (say)}$$
$$= 155 - 2\alpha.$$

Then if $\alpha = 30^\circ$, $\gamma_1 = 95^\circ$. Some designers make this angle 90°, others more, and still others less than that amount. Weisbach suggests that it be less so that the bucket will be shorter and friction less. This reasoning appears to be correct for the inflow wheel, for the size and conditions shown in Table II., but not for the outflow wheel. In the Tremont turbines, described in the *Lowell Hydraulic Experiments*, this angle is 90°, the angle α, 20°, and γ_2, 10°. Fourneyron made $\gamma_1 = 90^\circ$, and α from 30° to 33°.

In Table III. it appears that for $\gamma_1 = 150^\circ$, $V = 2.1\sqrt{gH}$, which exceeds $\sqrt{2gH}$; that is, the velocity of exit from the supply chamber exceeds that due to the head, which condition must result from a negative pressure at entrance into the wheel. For zero pressure for the frictionless wheel, the above condition gives

$$\gamma_1 = 180^\circ - 2\alpha,$$

which for $\alpha = 25°$, gives $\gamma_1 = 130°$, and for $\gamma_1 = 150°$, the pressure would be negative, and for 120° it would be positive. It appears that for the wheel *with friction*, considered in the table, that this pressure is also positive for $\gamma_1 = 120°$, and negative for 150°.

11. *Form of Bucket.*

The form of the bucket does not enter the analysis, and therefore its proper form cannot be determined analytically. Only the initial and terminal directions enter directly, and from these and the volume of the water flowing through the wheel, the area of the normal sections may be found from equations (20).

But well-known physical facts determine that the changes of curvature and section must be gradual, and the general form regular, so that eddies and whirls shall not be formed. For the same reason the wheel must be run with the correct velocity to secure the best effect; for otherwise the effective angles α and γ_1 may be changed to values which cannot be determined beforehand, in which case the wheel cannot be correctly ana-

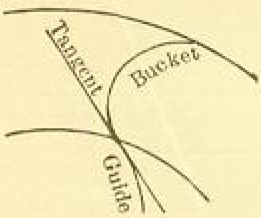

Fig. 5.

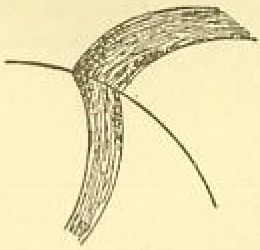
Fig. 6.

lyzed. In practice the buckets are made of two or three arcs of circles mutually tangential at their points of meeting. Also, if the normal sections, K, k_1, k_2, of the buckets as constructed do not agree with those given by computation, the stream will, if possible, adjust itself to true conditions by the formation of eddies. If the terminal sections at the guides, or the initial section of the bucket, be too small, the action may be changed from a *pressure wheel* to one of *free deviation*. So long as the pressure in the wheel exceeds the external pressure, the preceding analysis is applicable for the wheel running for best effect, observing that the sections K, k_1, k_2, are not those of the wheel, but those which are computed from the velocities V, v_1, v_2.

12. *Value of θ*; or direction of the quitting water.

From Fig. 1 it may be found that

$$w \cos \theta = v_2 \cos \gamma_2 - \omega' r_2, \quad . \quad . \quad . \quad . \quad . \quad (33)$$

and

$$w \sin \theta = v_2 \sin \gamma_2 ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

$$\therefore \cot \theta = \cot \gamma_2 - \frac{\omega' r_2}{v_2 \sin \gamma_2} . \quad . \quad . \quad . \quad . \quad (35)$$

These formulas are for the velocity giving maximum efficiency. If the speed be assumed, ω in place of ω' becoming known, v_2 is given by equation (19). It is apparent for such a case that θ may have a large range of values from $\theta = \gamma_2$, when the wheel is at rest, to θ exceeding 90° for high velocities. The following table gives some results :

TABLE IV.

$\alpha = 25°$, $\gamma_2 = 12°$, $\mu_1 = \mu_2 = 0.10$.

γ_1	$r_1 = r_2\sqrt{\tfrac{1}{2}}$.		$r_1 = 1.4\, r_2$	
	ω	θ	ω	θ
60°	$.314\sqrt{gH}$	72° 14′	$.160\sqrt{gH}$	103° 43′
90°	.310 "	66° 59′	.143 "	101° 17′
120°	.241 "	60° 24′	.126 "	82° 52′
150°	.157 "	55° 26′	.043 "	74° 51′

According to this table the water is thrown backward, or in the direction opposite to the motion of the wheel for the outward flow wheel, and for the inflow it is thrown forward for γ_1 less than 90°, and backward for γ_1 greater than 120°.

In the *Tremont turbine* a device was used for determining the direction of the water leaving the wheel, and for the best efficiency, 79¼ per cent., the angle θ was about 120°. *Lowell Hydraulic Experiments*, p. 33.

The angle thus observed had a large range of values ranging from 50° to 140° for efficiencies only two or three per cent. less than 79¼ per cent.

13. *Of the value of* ω.

So far as analysis indicates, the wheel may run at any speed; but in order that the stream shall flow smoothly from the supply chamber into the bucket—thus practically maintaining the angles α and γ_1—the relations in equations (5) and (6) must be maintained, or

$$v_1 = \frac{\sin \alpha}{\sin \gamma_1} V, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

and this requires that the velocity V shall be properly regulated, which can be done by regulating the head h_1 or the pressure p_1 or both h_1 and p_1, as shown by equation (4). This however is not practical. In practice, the speed is regulated, and when the condition for maximum efficiency is established, the velocities V and v_1 are found from equations (17) and (18).

Since γ_2, in practice, is small we have, for best effect,

$$v_2 = \omega' r_2, \text{ approximately}, \quad . \quad . \quad . \quad . \quad (37)$$

and, adopting this value, a more simple expression may be found for the velocity of the wheel. For equation (19) gives

$$v_2 = r_2 \omega'$$

$$= \frac{\sqrt{gH}}{\sqrt{\frac{\cos \alpha \sin \gamma_1}{\sin(\alpha + \gamma_1)}\left(\frac{r_1}{r_2}\right)^2 + \frac{1}{2}\mu_1 \frac{\sin^2 \gamma_1}{\sin^2(\alpha + \gamma_1)}\left(\frac{r_1}{r_2}\right)^2 + \frac{1}{2}\mu_2}} . \text{(Approx.)} \quad (38)$$

If $\mu_1 = \mu_2 = 0.10$, $r_2 \div r_1 = 1.40$, $\alpha = 25°$, $\gamma_1 = 90°$, the velocity of the *initial* rim for outward flow will be

$$\omega' r_1 = \frac{\sqrt{gH}}{\sqrt{1 + 0.159}} = 0.929 \sqrt{gH}.$$

The velocity due to the head would be

$$v_h = \sqrt{2gH} = 1.414 \sqrt{gH};$$

hence, the velocity of the initial rim should be about

$$\frac{0.928 \sqrt{gH}}{1.414 \sqrt{gH}} = 0.659 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

of the velocity due to the head.

For an inflow wheel in which $r_1^2 = 2r_2^2$, and the other dimensions, as given above, this becomes

$$\frac{0.954}{1.414} = 0.689 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

of the velocity due to the head.

The highest efficiency of the Tremont turbine, found experimentally, was 0.79375, and the corresponding velocity, 0.62645 of the velocity due to the head, and for all velocities above and below this value the efficiency was less. Experiment showed that the velocity might be considerably larger or smaller than this amount without diminishing the efficiency very much.

In the Tremont turbine it was found that if the velocity of the initial (or interior) rim was not less than 44 nor more than 75 per cent. of that due to the fall, the efficiency was 75 per cent. or more. *Exp.*, p. 44.

This wheel was allowed to run freely without any brake except its own friction, and the velocity of the initial rim was observed to be 1.335 $\sqrt{2gH}$, half of which is

$$0.6675 \sqrt{2gH}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

"which is not far from the velocity giving maximum effect; that is to say, *when the gate is fully raised the coefficient of effect is a maximum when the wheel is moving with about half its maximum velocity.*" *Exp.*, p. 37.

M. Poncelet computed the theoretical useful effect of a certain turbine of which M. Morin had determined the value by experiment. The following are the results (*Comptes Rendus*, 1838, *Juillet*):

TABLE V.

Velocity of initial rim or $\frac{r_1\omega'}{\sqrt{2gH}}=$	Number of turns of the wheel per minute.	Ratio of useful to theoretical effect.	Means of values by experiment.
0.0	0.00	0.000	....
0.2	33.80	0.664	....
0.4	47.87	0.773	0.700
0.6	58.61	0.807	0.705
0.7	62.81	0.810	0.700
0.8	67.67	0.806	0.675
1.0	75.76	0.786	0.610
1.2	82.88	0.753	0.490
1.4	89.52	0.712	0.360
1.6	95.70	0.664	0.280
1.8	101.51	0.612	0.203
2.0	107.00	0.546	0.050
3.72	145.00	0.000	....

Poncelet states that he took no account of passive resistances, and hence his results should be larger than those of experiment as they are; but here both theory and experiment give the maximum efficiency for a velocity of about 0.6 that due to the head, and the efficiency is but little less for velocities perceptibly greater and less than that for the best effect. For velocities considerably greater and less, theoretical results are much larger than those found by experiment, for reasons already given, chief of which is the fact that eddies are induced, and the effective angles of the mechanism changed to unknown values.

14. *Pressure in the wheel.*

Dropping the subscript $_2$ from v, r, p, in equation (9), the resulting value of p will give the pressure per unit at any point of the bucket providing that μ_2 be considered constant. Changing r to ρ, equation (9) thus gives

$$p = \left[v_1^2 - (1 + \mu_2)\, v^2 + \omega^2 (\rho^2 - r_1^2) \right] \frac{\delta}{2g} + p_1. \quad . \quad (42)$$

To solve this requires a knowledge of the transverse sections of the stream, for the velocity v will be inversely as the cross section.

From equations (20) and (6)

$$v = v_1 \frac{k_1}{k} = \frac{k_1}{k} \cdot \frac{\sin \alpha}{\sin (\alpha + \gamma_1)} \cdot \omega r_1 \quad . \quad . \quad . \quad (43)$$

From (4) and (5),

$$p_1 = \delta h_1 + p_a - \delta \frac{1 + \mu_1}{2g} \cdot \frac{\sin^2 \gamma_1}{\sin^2 (\alpha + \gamma_1)}\, \omega^2 r_1^2. \quad . \quad (44)$$

These reduce equation (42) to

$$\frac{gp}{\delta} = \frac{gp_a}{\delta} + gh_1 + \tfrac{1}{2}\, \omega^2 \rho^2 - \tfrac{1}{2}\, \omega^2 r_1^2$$

$$- \frac{1}{2} \left[(1 + \mu_1) \sin^2 \gamma_1 + [k_1^2 (1 + \mu_2) - k^2] \frac{\sin^2 \alpha}{k^2} \right] \frac{\omega^2 r_1^2}{\sin^2 (\alpha + \gamma_1)}. \, (45)$$

The back or concave side of the bucket will be subjected to a pressure which may be considered in two parts: one due to the deflection of the stream passing through it, the other to a pressure which is the same as that against the crown, and is uniform throughout the cross section of the bucket, due to the pressure of a part (or all) of the head in the supply chamber. It is the latter pressure which is given by the value of p in equation (45). The construction of the wheel being known, the pressure p may be found at any point of the wheel for any assumed practical velocity; although, for reasons previously given, it will be of practical value only when running near the velocity for maximum efficiency. There are two cases:

1. That in which the discharge is into free air;
2. That in which the wheel is submerged.

In the first case if the pressure is uniform, the case is called that of "free deviation" in which the entire pressure upon the forward side of the bucket is due to the deviation of the water from a right line, and will be considered further on.

If equation (45) shows a continually decreasing pressure from the initial element to that of exit, or if the minimum pressure exceeds p_a, the preceding analysis is applicable. But if it shows a point of minimum pressure less than p_a, it will be in a condition of unstable equilibrium, in which the slightest inequality would cause air to rush in and restore the pressure to that of the atmosphere; so that the pressure in the wheel and the flow would be changed. The point of minimum pressure may be found by plotting results found from equation (45), substituting values for ρ taken from measurements of the wheel, and k from computation. From the entrance of the wheel up to the point of minimum pressure the preceding analysis applies; and the remainder of the wheel must be analyzed for "free deviation" and the two results added.

In the second case the equations will apply, since air cannot enter, provided that p does not become negative, to realize which requires a tensile stress of the water. This is impossible and eddies would be formed; and the effect of these on the velocity and pressure cannot be computed. Such a case cannot be analyzed.

15. *To find the pressure at the entrance to the bucket* when running at best effect. In (45) let $\rho = r_1$, $k = k_1$ and $p = p_1$. To simplify still more, let the wheel be frictionless, or $\mu_1 = \mu_2 = 0$, and find from equation (38)

$$r_1^2\omega^2 = \frac{\sin(\alpha + \gamma_1)}{\cos\alpha \sin\gamma_1} gH, \quad . \quad . \quad . \quad . \quad . \quad (46)$$

also $h_1 = H + h_2$, and (45) becomes

$$p_1 = \delta H + \delta h_2 + p_a - \frac{\delta \sin \gamma_1}{2 \cos \alpha \sin (\alpha + \gamma_1)} H. \quad . \quad (47)$$

If the wheel is not submerged $h_2 = 0$, and let the pressure p_1 equal that of the atmosphere, or p_a, then

$$0 = 1 - \frac{\sin \gamma_1}{2 \cos \alpha \sin (\alpha + \gamma_1)}. \quad . \quad . \quad . \quad . \quad (48)$$

If the wheel be submerged, let $p_1 = \delta h_2 + p_a$, and the equation reduces to that of the preceding.

Equation (48) gives

$$\tan 2\alpha = - \tan \gamma_1,$$

or,

$$2\alpha = 180° - \gamma_1; \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

for which value the pressure at the entrance to the wheel will equal that just outside.

If,

$$2\alpha > 180° - \gamma_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

the pressure within will be less than that without; but if

$$2\alpha < 180° - \gamma_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

the pressure within will exceed that without—a condition which is considered desirable. If frictional resistances be considered the value of $r_1\omega'$ from equation (38) will be less than that given by (46), and hence the last term of equation (47) will be less unless α be greater than the value given by equation (51); hence with frictional resistances the terminal angle of the guide blade may exceed somewhat $90° - \frac{1}{2}\gamma_1$; therefore, if the value of α be found for a frictionless wheel it will be a safe value when there is friction. If $\gamma = 90°$ and $\alpha = 90° - \frac{1}{2}\gamma_1 = 45°$, then (47) gives

$$p_1 = \delta H + \delta h_2 + p_a - \delta H = \delta h_2 + p_a, \quad . \quad . \quad (52)$$

as it should. If $\gamma_1 = 90°$ and $\alpha = 30°$, then

$$p_1 = \delta H + \delta h_2 + p_a - \tfrac{2}{3}\delta H,$$

or, $$p_1 > \delta h_2 + p_a + 0.33\delta H. \quad (53)$$

The angle α should not be so large or γ_1 so small as to produce excessive pressure at the entrance to the wheel.

Example.—Find the pressure per square inch at the entrance to the wheel when the head is 10 feet, the terminal angle of the guide is 30°, the initial angle of the bucket $\gamma_1 = 90°$; the wheel being one foot under the water in the tail race.

16. *Number of buckets.*

The analysis given above is true for a wheel with a single bucket, provided the supply is constantly open to the bucket and closed by the remainder of the wheel. But for practical considerations the wheel should be full of buckets, although the number cannot be determined by analysis. Successful wheels have been made in which the distance between the buckets was as small as 0.75 of an inch, and others as much as 2.75 inches. *Lowell Hyd. Exp.*, p. 47. Turbines at the Centennial Exposition had buckets from 4 inches to 9 inches from centre to centre.

17. *Ratio of radii.*

Theory does not limit the dimensions of the wheel. In practice,

$$\left.\begin{array}{l}\text{for outward flow, } r_2 \div r_1 \text{ is from 1.25 to 1.50}\\ \text{for inward flow, } r_2 \div r_1 \text{ is from 0.66 to 0.80}\end{array}\right\} \quad (54)$$

It appears from Table II. that the inflow wheel has a higher efficiency than the outward flow wheel (columns 6 and 3), and these wheels have about the same outside and inside diameters. The inflow wheel also runs somewhat slower for best effect. The centrifugal force in the outward flow wheel tends to force the water outward faster than it would otherwise flow; while in the inward flow wheel it has the contrary effect, acting as it does in opposition to the velocity in the buckets.

It also appears that the efficiency of the outward flow wheel increases slightly as the width of the crown is less, and the velocity for maximum efficiency is slower; while for the inflow wheel the efficiency slightly increases for increased width of crown and the velocity of the outer rim at the same time also increases.

Let $r_1 = nr_2$, $\gamma_1 = 90^\circ$, $\gamma_2 = 20^\circ$, $\mu_1 = \mu_2 = 0$;

then for $n = 0$, 0.5, 0.8, 1, 1.4, $\omega r_1 = 2$, 3,

will $\omega r_2 = \infty$

and $E =$

18. *Efficiency*, E.

The method of determining the theoretical value of E has already been given; but to determine the actual value, resort must be had to experiments. These have been made in large numbers and the results published. By assuming the minimum values of the several losses, a maximum limit to the efficiency may be fixed. Thus, if the actual velocity be 0.97 of the theoretical, the energy lost will be $(1 - 0.97^2)$ or 6 per cent.

Friction along the buckets and bends . . .	5	"	"
Energy lost by impact, say	2	"	"
Energy lost in the escaping water	3	"	"
Total	16	"	"
Leaving	84	"	"

available for work. This discards the friction of the mechanism and frictional losses along the guides, and if 2 per cent. be allowed for the latter, there will be left 82 per cent. It seems hardly possible for the effective efficiency to exceed 82 per cent., and all claims of 90 or more per cent. for these motors should be at once discarded as being too improbable for serious consideration. A turbine yielding from 75 to 80 per cent. is extremely good. The celebrated Tremont turbine gave 79¼ per cent. *Lowell Exp.*, p. 33. Experiments with higher efficiencies have been reported. A Jonval turbine (parallel flow) was reported as yielding 0.75 to 0.90, but Morin suggested corrections reducing it to 0.63 to 0.71. (Weisbach, *Mech. of Eng.*, vol. ii., p. 501.) Weisbach gives the results of many experiments, in which the efficiency ranged from 50 to 84 per cent. See pages 470, 500–507. See also *Jour. Frank. Inst.*, 1843, for efficiencies from 64 to 75 per cent. Numerous experiments give E = 0.60 to 0.65. The efficiency, considering only the energy imparted to the wheel, will exceed by several per cent. the efficiency of the wheel, for the latter will include the friction of the support, and leakage at the joint between the sluice and wheel, which are not included in the former; also as a plant the resistances and losses in the supply chamber are to be still further deducted.

19. *The Crowns.*—The crowns may be plane annular discs, or conical, or curved. If the partitions forming the buckets be so thin that they may be discarded, the law of radial flow will be determined by the form of the crowns. If the crowns be plane, the radial flow (or radial component) will diminish as the distance from the axis increases—the buckets being full—for the annular space will be greater.

20. *Designing.*

The dimensions of a wheel must be determined for a definite velocity. Thus far it has been assumed that the angles α, γ_1, etc., are given, and the normal sections of the stream thus deduced. We will now assume that all the dimensions of the buckets are known, and the angle α and the section K are to be determined. The velocities v_1 and v_2 must now be found independently of α. From Fig. 1 we have

$$V^2 = v_1^2 + \omega^2 r_1^2 - 2v_1\omega r_1 \cos \gamma_1 \quad . \quad . \quad . \quad (55)$$

which combined with equations (4), (9), (10), $v_1 k_1 = v_2 k_2$, as in (20), and $H = h_1 - h_2$, will give

$$v_2 = \frac{k_1}{k_2} v_1 = \frac{(1 + \mu_1) k_1 k_2 \omega r_1 \cos \gamma_1}{(1 + \mu_2) k_1^2 + \mu_1 k_2^2}$$

$$+ \sqrt{\frac{2gH + [r_2^2 - (2 + \mu_1) r_1^2] \omega^2}{(1 + \mu_2) k_1^2 + \mu_1 k_2^2} k_1^2 + \left[\frac{(1 + \mu_1) k_1 k_2 \omega r_1 \cos \gamma_1}{(1 + \mu_2) k_1^2 + \mu_1 k_2^2}\right]^2}$$

$$= P\omega + \sqrt{2gHQ + R^2\omega^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

where

$$P = \frac{(1 + \mu_1) k_1 k_2 r_1 \cos \gamma_1}{(1 + \mu_2) k_1^2 + \mu_1 k_2^2}, \quad . \quad . \quad . \quad . \quad (56a)$$

$$Q = \frac{k_1^2}{(1 + \mu_2) k_1^2 + \mu_1 k_2^2} \quad . \quad . \quad . \quad . \quad . \quad (56b)$$

$$R^2 = \frac{r_2^2 - (2 + \mu_1) r_1^2}{(1 + \mu_2) k_1^2 + \mu_1 k_2^2} k_1^2 + P^2 \quad . \quad . \quad . \quad (56c)$$

$$v_1 = \frac{k_2}{k_1} v_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

Equation (11) gives the velocity of exit. Equation (14), or (130), gives

$$E = \frac{U}{\delta QH} = \frac{1}{gH}\left[(r_1^2 - r_2^2)\,\omega^2 + \left(r_2 \cos \gamma_2 - \frac{k_1}{k_2} r_1 \cos \gamma_1\right) v_2 \omega\right]$$

$$= \frac{1}{gH}[- S^2\omega^2 + Tv_2\omega] \quad . \quad . \quad . \quad . \quad . \quad (58)$$

in which

$$S^2 = r_2^2 - r_1^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58a)$$

$$T = r_2 \cos \gamma_2 - \frac{k_1}{k_2} r_1 \cos \gamma_1. \quad . \quad . \quad . \quad . \quad (58b)$$

Substituting v_2 from equation (56) gives

$$E = \frac{1}{gH}\left[(-S^2 + TP)\,\omega^2 + T\sqrt{2gHQ\omega^2 + R^2\omega^4}\right]$$

$$= \frac{1}{gH}\left[-V^2\omega^2 + T\sqrt{2gHQ\omega^2\ R^2\omega^4}\right]. \quad . \quad . \quad (59)$$

in which

$$V^2 = TP - S^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59a)$$

For E a maximum, make $\frac{dE}{d\omega} = 0$, giving

$$\omega V^2\sqrt{2gHQ + R^2\omega^2} = T(gHQ + R^2\omega^2);$$

$$\therefore \omega' = \frac{\sqrt{gHQ}}{R}\sqrt{\frac{V^2 - \sqrt{V^4 - T^2R^2}}{\sqrt{V^4 - T^2R^2}}} \quad . \quad . \quad . (60)$$

which is of the same form as equation (16). This value of ω' substituted in equation (59) will give the maximum efficiency, or

$$E_{\text{max.}} = (V^2 - \sqrt{V^4 - T^2R^2})\,\frac{Q}{R^2}. \quad . \quad . \quad . \quad (61)$$

For the frictionless wheel $Q = 1$.

To find the terminable angle α of the guide blade that will enable the stream to flow smoothly, subject to the preceding conditions, Fig. 1 gives

$$V\cos\alpha = \omega r_1 - v_1\cos\gamma_1,$$

which, combined with equation (55), gives

$$\cos\alpha = \frac{\omega r_1 - v_1\cos\gamma_1}{\sqrt{v_1^2 + \omega^2 r_1^2 - 2v_1\omega r_1\cos\gamma_1}}. \quad . \quad . \quad (62)$$

Eliminating v_1 by means of equations (57) and (56) gives $\cos\alpha$ in terms of the six constants r_1, r_2, γ_1, γ_2, k_1 and k_2, which are fixed and known from the dimensions of the wheel, and of the velocity ω of the wheel. Since the wheel may run at different velocities the angle α must vary, and this will be done in practice by the piling of the water in the passages. Each turbine, however, should be designed to run at the speed giving maximum efficiency, and its angles and dimensions should satisfy equations (60) and (62).

From equation (9),

$$2g\frac{p_1}{\delta} = (1 + \mu_2)\,v^2_2 - v^2_1 - \omega^2\,(r^2_2 - r^2_1) + 2g\,\frac{p_a + \delta h_2}{\delta}, \quad . \quad (63)$$

in which, if v_2 and v_1 be substituted from above, p_1 becomes known.

Similarly, V from equation (55) becomes known, and finally, from (60),

$$\cos\alpha = \frac{\omega r_1 - v_1\cos\gamma_1}{V}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

21. *Path of the Water.*—Let aA be the position of the bucket when the water enters at b. The bucket being drawn in position to a scale, divide it into any number of parts—equal or unequal—aa_1, a_1a_2, etc., and find the time required for it to go from a to a_1. The distance being small, assume that the velocity is uniform from a to a_1, and equal to v, which will be given by equation (56) by dropping all the subscripts $_2$ and changing r to ρ, thus,

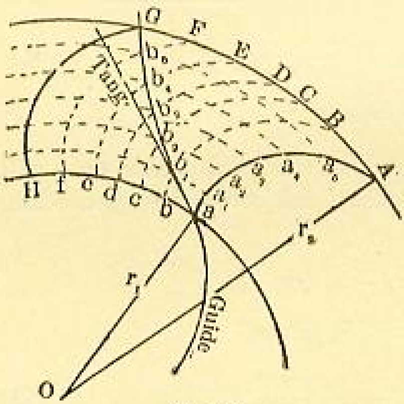

Fig. 7.

$$v = \frac{(1 + \mu_1)\,k_1\,k\omega r_1\cos\gamma_1}{(1 + \mu_2)\,k_1^2 + \mu_1\,k^2}$$

$$+ \sqrt{\frac{2gH + \omega^2\,[\rho^2 - (2 + \mu_1)\,r_1^2]}{(1 + \mu_2)\,k_1^2 + \mu_1\,k^2}k_1^2 + \left[\frac{(1 + \mu_1)\,k_1\,k\omega r_1\cos\gamma_1}{(1 + \mu_2)\,k_1^2 + \mu_1\,k_2^2}\right]^2}. \quad (64a)$$

Then will the time t be

$$t = \frac{aa_1}{v} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

During this time the rim has gone from a to b a distance

$$ab = r_1 \omega t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

If the bucket bB be drawn through b, and the arc $a_1 b_1$ through a_1, their intersection b_1 will be the position of the particle at the end of the time t. In a similar manner, the successive points b_2, b_3, etc., may be found, through which a continuous curve may be drawn representing the path of the stream.

The line tangent to the termination of the bucket, will indicate the direction of the water at entrance of the wheel, and if the water drives the wheel, the *path* should be entirely outside this line and convex toward it.

22. *Design a guide blade, outflow turbine.*
Assume the effective fall, ... $H =$ ft.
Assume the required horse-power, . . . $HP =$
Assume the exit angle, ... $\gamma_2 =$
Assume the entrance angle of bucket, . . $\gamma_1 = 90°$.
Fix the exit angle of the guide, Eqs. (32), (51), $\alpha = 30°$. ?
Assume efficiency, ... $E = 0.65$, or **0.70**;
and after the wheel is fully designed, recompute this value and if necessary correct the dimensions.
Required quantity of water per second without loss, ... $Q = U \div \delta H$.
Required quantity, ... $Q \div E =$
Assume $\mu_1 = 0.10$, $\mu_2 = 0.075$.
Velocity of the initial rim, Eq. (40), approx., $\omega r_1 =$
(The corrected, final value will be found by Eq. (16) or (60).

Let $r_2 = 1.3r_1$, then velocity of outer rim, $\omega r_2 =$
The velocity into the bucket, Eq. (18), . . $v_2 =$
Initial section of buckets, Eq. (20), . . . $k_1 =$
The inner circumference will be $2\pi r_1$. Let the walls of the buckets be $\frac{1}{16}$ of the circumference, then the effective openings will be $\frac{15}{16}$ of the circumference, or $\frac{15}{8}\pi r_1$. Assume a depth, y, between the crowns. Try $y = \frac{1}{2}r_1$. Then will the initial cross section of all the buckets be $\frac{15}{16}\pi r_1^2$; hence, $\frac{15}{16}\pi r_1^2 = k_1$; $\therefore$ $r_1 =$
If the radius is not what is desired, it may be changed to some other value and the depth y computed. Then, . . . $r_2 =$
The cross section at outer rim will be, if the crowns are planes, $\frac{15}{8}\pi r_2\, y \sin \gamma_2 = k_2 =$
The number of buckets will be assumed . =

Having determined these elements, the final velocity v_2 in reference to the bucket may be computed by equation (67), ωr_1 from (62), V from (55), and α from (64).

If the turbine revolves in air, at least half the depth of the wheel is to be deducted from the head H.

If the circular opening between the wheel and sluice be $\frac{1}{16}$ of an inch, or $\frac{1}{192}$ of a foot, and the coefficient of discharge be 0.7, the discharge will be

$$\frac{1}{192}\, 0.7 \times 2\pi r_1 \times \sqrt{\frac{2gp_1}{\delta}} = q, \quad . \quad . \quad . \quad . \quad (67)$$

p_1 being determined from equation (63). The loss of work will be

$$62.2q \times H \quad \text{or} \quad 62.2q\,(H - \tfrac{1}{2}y). \quad . \quad . \quad . \quad . \quad (68)$$

The work lost by friction, if the radius of the axle be r_3, the weight of the loaded wheel, W_3, and coefficient of friction μ_3, will be nearly

$$\frac{2}{3}\mu_3 W_3 r_3 \omega \text{ per sec.} \quad . \quad . \quad . \quad . \quad . \quad (69)$$

The work done by the water must be the effective U plus the work due to the losses. The work done by the water passing through the wheel must be U plus that given by equation (69). Call this U_1. Compute the work done by the water, and let it be U_2; then will the required depth be

$$y_2 = \frac{U_1}{U_2} y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

The efficiency may now be recomputed.

SPECIAL WHEELS.

23. *Fourneyron Turbine.*—All wheels having guide blades are of the Fourneyron type, although the wheels made by him were outward flow. The preceding analysis is a general solution of this turbine.

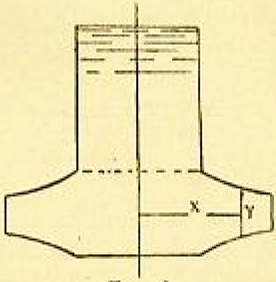

FIG. 8.

24. *Francis and Thomson's vortex wheels* are inward flow wheels with guide blades. The preceding analysis is also applicable to these wheels.

25. *The Jonval Turbine* is a parallel flow wheel with guide blades, to which the preceding analysis is applicable by making $r_1 = r_2$.

(For the details of these and many other forms, see *Hydraulic Motors* by Weisbach.)

26. *Rankine Wheel.*—This is a wheel of the Fourneyron type, but Rankine having made certain modifications in its

assumed construction it is indicated by his own name. (Fig. 8.)

It is an outflow wheel and the crowns are so made that the *radial* velocity of the water in passing through the wheel will be uniform. If x be the abscissa from the axis of the wheel to any point of the crown, and y the distance between the crowns at that point, v_r the radial component of the velocity, then

$$y \cdot 2\pi x \cdot v_r = Q,$$

or,

$$yx = a \text{ constant}, \quad . \quad . \quad . \quad . \quad (71)$$

which is the equation to an hyperbola referred to its asymptotes. This determines the form of the crowns. If the wheel were inward flow, the depth would be greatest at the inner rim.

In this wheel the initial element of the bucket is radial, or $\gamma_1 = 90°$; and Rankine *assumed* that the velocity for best effect must be such that the water will quit the wheel radially, or $\theta = 90°$. These conditions given, from Fig. 1 and equations (5), (6), (34), for a *frictionless* wheel,

$$\omega r_1 = V \cos \alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

$$\omega r_2 = v_2 \cos \gamma_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

$$v_1 = w = V \sin \alpha = \omega r_1 \tan \alpha = \omega r_2 \tan \gamma_2 = v_2 \sin \gamma_2; \quad (74)$$

$$\therefore \tan \alpha = \frac{r_2}{r_1} \tan \gamma_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

which determines the proper angle of the guide blade when the value of γ_2 has been assigned. If $\gamma_2 = 15°$, $r_2 = 1\frac{1}{4} r_1$, then $\alpha = 18\frac{1}{2}°$; and if $\gamma_2 = 20$, $r_2 = 1.3$, then $\alpha = 24°$ nearly. But to be certain that the internal pressure exceeds the external, α should exceed these values. Equations (19), (73), and (75) give

$$\omega r_1 = \sqrt{\frac{2gH}{2 + \tan^2 \alpha}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

which establishes the velocity of the initial rim of the wheel.

The work in the frictionless wheel will be the theoretical work the water could do, less the energy in the water quitting the wheel, or

$$U = WH - \tfrac{1}{2}\frac{W}{g}w^2$$

[Eq. (74)] $$= WH - \tfrac{1}{2}\frac{W}{g}r_1^2\omega^2 \tan^2 \alpha. \quad . \quad . \quad . \quad (77)$$

[Eq. (76)] $$= \frac{WH}{1+\frac{1}{2}\tan^2\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

The efficiency will be

$$E = \frac{U}{WH} = \frac{1}{1+\frac{1}{2}\tan^2\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

27. The following is to show that Rankine's assumption of velocity for best efficiency is not quite correct. Substituting $\gamma_1 = 90^\circ$, $\mu_1 = 0$, $\mu_2 = 0$, $r_1 = nr_2$, in equation (16) gives

$$\omega^2 r^2_2 = \frac{gH}{1-2n^2}\left[\frac{1-n^2-\sqrt{(1-n^2)^2-\cos^2\gamma_2(1-2n^2)}}{\sqrt{(1-n^2)^2-\cos^2\gamma_2(1-2n^2)}}\right] \quad (80)$$

and these in equation (19) will give

$$v_2\cos\gamma_2 = \sqrt{gH\cos^2\gamma_2\left[\frac{1-n^2+\sqrt{(1-n^2)^2-\cos^2\gamma_2(1-2n^2)}}{\sqrt{(1-n^2)^2-\cos^2\gamma_2(1-2n^2)}}\right]}$$

$$= \omega r_2\left[1-n^2+\sqrt{(1-n^2)^2-\cos^2\gamma_2(1-2n^2)}\right]. \quad . \quad (81)$$

This does not give

$$v_2 \cos \gamma_2 = \omega r_2$$

as in equation (73), except when $n^2 = \frac{1}{2}$ (or $2r^2_1 = r^2_2$); and hence the direction of the water at exit will not be radial, as Rankine assumed, except for this case. In practice, outflow wheels are constructed almost exactly with this proportion, $n^2 = \frac{1}{2}$, and hence the analysis from equations (72) to (79) is sufficiently exact for the frictionless outflow wheel; and, as seen above, the hypothesis greatly simplifies the analysis.

The condition for best efficiency of the frictionless wheel requires that the velocity of leaving the wheel should be a minimum; and this may be realized, in some cases, when its direction is oblique to the radius.

Thus, let AC be radial when AB is the velocity relative to the bucket, and BC the velocity of the rim; then it may be, in some cases, that when AD is the relative velocity of exit, AE, the velocity of exit relative to the earth, will be less than AC, as shown in Fig. 9.

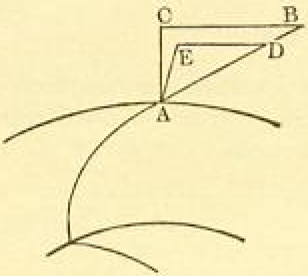

FIG. 9.

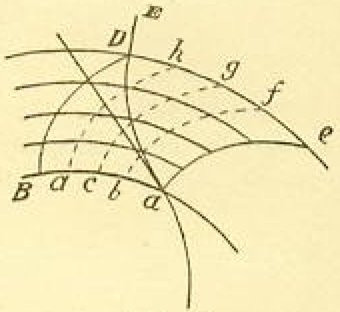

FIG. 10.

28. *The path of the water* is easily constructed for this wheel. Since the radial velocity is uniform, the time of flowing through the wheel will be

$$t = \frac{r_2 - r_1}{v_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

during which time the initial rim AB, Fig. 10, will have travelled

$$aB = \omega r_1 t \text{ feet.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

Divide $r_2 - r_1$ into equal parts by concentric arcs, and the space aB into the same number of equal parts, and through the points of division a, b, c, d, trace the buckets; then will aD, drawn through the proper intersections of the arcs and buckets, be the required path.

9. Analyze a Rankine turbine, having given: $H = 12$ feet, $\gamma_2 = 15^\circ$, $r_1 = 2$ feet, $r^2_2 = 2r^2_1$. Depth of outer rim, 6 inches.

Find Radius of outer rim, ... $r_2 =$
Angular velocity, ... $\omega =$
Velocity of initial rim, ... $r_1\omega =$
Velocity of outer rim, ... $r_2\omega =$
Angle of guide plates, ... $\alpha =$
Velocity from the supply chamber, . $V =$
Initial velocity in bucket, ... $v_1 =$
Terminal velocity in bucket, ... $v_2 =$
Velocity of quitting water, ... $w =$
Depth of inner rim, ... $y_1 =$
The horse-power, ... $HP =$
The efficiency, ... $E =$

If the partitions for the buckets occupy $\frac{1}{20}$ of the wheel, and the losses due to frictional resistances in the wheel and friction of the wheel be 20 per cent., what will be

The horse-power, ... $HP =$
The efficiency, ... $E =$

Find the pressure at the inner rim, . . . $p_1 =$
Find the path of the water.

29. *Velocity of a particle along a tube rotating about an axis perpendicular to its plane.*

This problem has already been solved in establishing the general equations of turbines, and the following is given to present it from another point of view.

If the particle at A, whose mass is m, be confined while the tube rotates about O, Fig. 11, with the angular velocity ω, the centrifugal force would be

$$f = m\omega^2\rho. \qquad \qquad (84)$$

If θ be the angle between the radius vector prolonged and the normal upon the tangent, the component in the direction of the tangent to the tube will be

$$m\omega^2\rho \sin \theta,$$

and when the particle is free to move, this component will be effective for producing motion, and if the pressures at the opposite ends of the element are not equal, but differ by an amount dp, we have the equation

$$m\frac{d^2s}{dt^2} = m\omega^2\rho \sin \theta - dp. \quad . \quad . \quad . \quad . \quad . \quad (85)$$

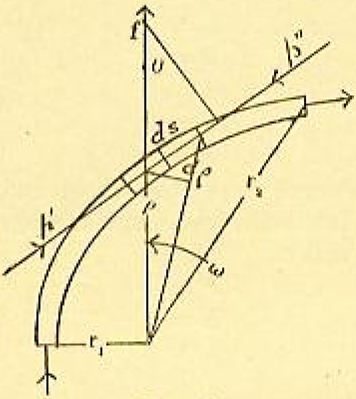

FIG. 11.

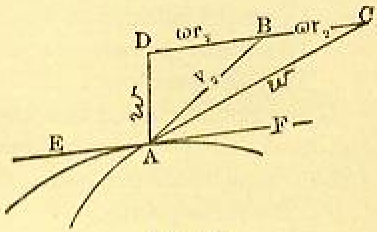

FIG. 12.

But $$ds \sin \theta = d\rho,$$

which combined with the preceding equation gives

$$\frac{dsd^2s}{dt^2} = \omega^2\rho d\rho - \frac{1}{m}\frac{d\rho}{\sin\theta}dp. \quad . \quad . \quad . \quad . \quad (86)$$

But $\frac{d\rho}{\sin \theta} = ds$, and if δ be the weight of unity of volume, then $m = \frac{\delta}{g}ds$, and the last term becomes $\frac{g}{\delta}dp$. The integral will be

$$\left.\frac{ds^2}{dt^2}\right]_{v_1}^{v_2} = v_2^2 - v_1^2 = \omega^2(r_2^2 - r_1^2) - 2g\frac{p_2 - p_1}{\delta}. \quad . \quad (87)$$

If the friction be $\mu_2 v_2^2$, the equation becomes

$$(1 + \mu_2) v_2^2 = v_1^2 + \omega^2 (r_2^2 - r_1^2) - 2g \frac{p_2 - p_1}{\delta}, \quad . \quad (88)$$

which is the same as equation (9).

If $p_2 = p_1$, and $\mu_2 = 0$, then

$$v_2^2 = v_1^2 + \omega^2 (r_2^2 - r_1^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

This gives the velocity relative to the tube whether it revolves to the right or left, and whatever be its curvature. If it revolves to the left, the resultant velocity will be AD, Fig. 12; if to the right, it will be AC. If γ_2 be measured from the arc backward of the motion, or $\gamma_2 = BAF$ for rotation to the left, and $\gamma_2 = EAB$ for motion to the right; then

$$AD^2 = w^2 = v_2^2 + \omega^2 r_2^2 - 2v_2 . \omega r_2 \cos \gamma_2. \quad . \quad . \quad (90)$$

$$AC^2 = w^2 = v_2^2 + \omega^2 r_2^2 + 2v_2 . \omega r_2 \cos \gamma_2. \quad . \quad . \quad (91)$$

In the latter case the quitting velocity will exceed the terminal velocity in the tube, and therefore increased velocity will have been imparted to the water—a condition requiring that energy be imparted to the wheel from an external source. In the former case the wheel is a motor, in the latter it is a receiver or transmitter of power; in the former the water drives the wheel, in the latter the wheel drives the water and virtually becomes a centrifugal pump.

If the water issues tangentially to the path described by the orifice, then $\gamma_2 = 0$, and

$$w = v_2 \mp \omega r_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

the upper sign belonging to the motor, and the lower to the pump.

Exercise.—If $r_1 = 1$ ft. $r_2 = 5$ ft. $\mu_2 = 0.1$, $v_1 = 5$ ft. per second, and the bucket rotates about a vertical axis 30 times per minute, and discharges the water directly backward, making $\gamma_2 = 0$, required the terminal velocity along the tube and the velocity of discharge relatively to the earth

30. *Wheel of Free Deviation.*—In this wheel the water in the buckets has a free surface, or, in other words, is subjected only to the pressure of the atmosphere. For this case

$$p_2 = p_1 = p_a\,;\ h_1 = H, \text{ and } h_2 = 0,$$

and equation (4) gives

$$(1 + \mu_1)\ V^2 = 2gH, \quad . \quad . \quad . \quad . \quad . \quad (93)$$

which will be the velocity of discharge from the supply chamber into the wheel; it is the velocity due to the head in the supply chamber when frictional resistance is included.

The triangle ABC, Fig. 1, gives

$$v_1^2 = V^2 + \omega^2 r_1^2 - 2\,V\omega r_1 \cos \alpha, \quad . \quad . \quad . \quad (94)$$

which substituted in equation (88) gives

$$(1 + \mu_2)\,v_2^2 = V^2 + \omega^2 r_2^2 - 2\,V.\,\omega r_1 \cos \alpha, \quad . \quad (95)$$

and this in equation (90) gives w^2, and equation (12) will give the required work. Equation (14) will give the velocity for best effect. But this involves a long analysis, and the following approximate solution is sufficiently accurate.

If γ_2 be small, and the wheel be run for best effect, that is, so as to make the velocity w very small, and considering $w = 0$, equation (92) makes

$$v_2 = \omega r_2 \text{ nearly.}$$

Using this value as if it were the exact one, also neglecting friction, (95) gives

$$2V.\,\omega r_1 \cos \alpha = V^2 = 2gH,$$

or

$$2\omega r_1 \cos \alpha = \sqrt{2gH}\,;$$

$$\therefore \omega r_1 = \frac{\sqrt{\tfrac{1}{2}gH}}{\cos \alpha} \cdot \text{(approx.)} \quad . \quad . \quad . \quad . \quad (96)$$

which gives the proper velocity of the initial rim ; and for the terminal rim

$$\omega r_2 = \frac{\sqrt{\frac{1}{2}gH}}{\cos\alpha}\cdot\frac{r_2}{r_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (97)$$

Number of revolutions per minute

$$N = 30\,\frac{\omega}{\pi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (98)$$

To find the velocity at any point of the bucket relative to the bucket, drop the subscript $_2$ from equation (95) giving

$$(1+\mu_2)\,v^2 = 2gH + \omega^2 r_2^2 - 2\sqrt{2gH}\,.\,\omega r_1 \cos\alpha. \quad . \qquad (99)$$

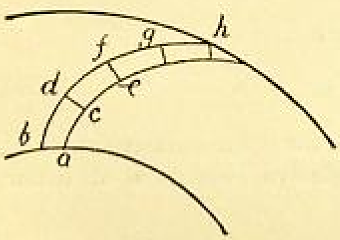

Fig. 13.

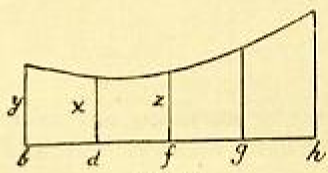

Fig. 14.

From Fig. 1 or equations (4) and (5) find

$$\sin\gamma_1 = \frac{\sqrt{2gH}}{v_1}\sin\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (100)$$

In the frictionless wheel, the work done will be

$$U = \tfrac{1}{2}M\,(V^2 - w^2), \quad . \quad . \quad . \quad . \quad . \quad . \qquad (101)$$

and the efficiency will be

$$E = \frac{U}{\delta QH}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (102)$$

31. *To find the form of the free surface*, let the bucket be very narrow, so that a normal to one of the curves will be approximately normal to the other. Divide one side of the bucket

into any convenient number of parts, as ac, ce, etc., and erect normals to the arc, as ab, cd, etc. Lay off these arcs on a right line. Compute the velocity at any point, as d, Fig. 13, by formula (99). Let x be the required depth at d, then because the velocity into the section equals q, the volume passing through one of the buckets per second, we have

$$x . dc . v = q;$$

$$\therefore x = \frac{q}{dc . v}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103)$$

and similarly for all other sections. If only relative heights are to be found, the quantity q need not be found, for if y be the height at b, Fig. 14, then

$$y . ba . v_1 = q;$$

$$\therefore x = \frac{ba . v_1}{dc . v} y, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

and by assuming any arbitrary value for y the relative value of x becomes known. Similarly, the relative heights at all other sections may be found.

32. *To find the path of the fluid* in reference to the earth, proceed as in Article 21 of the discussion of the general case.

33. *Exercise.*—Design a 30 horse-power inflow turbine of free deviation, given an effective head of 16 feet.

Assume the depth of gate opening to be 4 inches ($\frac{1}{3}$ foot), and after the computation has been completed if it does not give 30 horse-power the depth may be changed by proportion. Let the radius of the outer or initial rim be 1 ft.; of the inner rim, $\frac{2}{3}$ of a foot; terminal angle of the bucket, $\gamma_2 = 15°$; terminal angle of the guide, $\alpha = 30°$, $\mu_1 = 0.10 = \mu_2$.

Then, velocity of exit from supply chamber,

Eq. (93), ... $V =$

Velocity of outer rim, Eq. (96), ... $\omega r_1 =$

Velocity of inner rim, Eq. (97), ... $\omega r_2 =$

Number of turns per minute, ... =
Initial angle of bucket, Eq. (100), ... $\gamma_1 =$
Initial velocity in bucket, Eq. (94), ... $v_1 =$
Terminal velocity in bucket, Eq. (95), . . . $v_2 =$
Velocity of exit, Eq. (90), ... $w =$
Direction of outflow, Eq. (35), ... $\theta =$
Coefficient of discharge 0.60, volume of water, ... $Q =$
Weight of water ($\delta = 62.4$), ... $\delta Q =$
Work per second, Eq. (101), ... $U =$
Horse-power, ... $HP =$
Efficiency, ... $E =$

If 90 per cent. of U is effective work, and if this does not give 30 horse-power, then the depth of the wheel should be

$$d = \frac{30}{0.9U} 4 \text{ inches.}$$

Find the profile of the stream in the buckets.

34. The following is taken from the report of the Commissioners of the Centennial Exposition, 1876, on Turbines, Group XX. The tests were for two minutes each. The revolutions and horse-powers here given are those corresponding to the best efficiencies:

Diameter of wheel. Inches.	Head in supply chamber. Feet.	Revolutions per minute.	Horse-power.	Efficiency, per cent.	No. of Buckets.	Kind of wheel.
30	31	255	95	85.0	10	Inflow.
24	31	302	67	77.0	14	Parallel.
24	30½	310	64	74.5	13	
27	30	291	76.8	80.3	16	
30	30	257	74	75.5	18	
25	31	288	46	82.0	12	Parallel.
30	29.2	258	80.5	78.7	13	In and down
25	30	279	62.5	83.7	15	In and down.
27	30.4	246	53.2	73.6	14	Parallel.
36	29.6.	197	66.3	83.8	26	Parallel.

These tests were by no means exhaustive. It is not known that they were run for best effect. The distance from centre to centre of buckets varied from 4.3 inches to 9.5, and at these extreme values the efficiencies were about the same. The number of gate openings was less than the number of buckets.

TURBINES WITHOUT GUIDES.

35. *Barker's Mill.*—As ordinarily constructed, this motor has two hollow arms connected with a central supply chamber, with orifices near their outer ends and on opposite sides of the arms. There are no guide plates The supply chamber rotates with the arms. The arms may be cylindrical, conical, or other convenient shape.

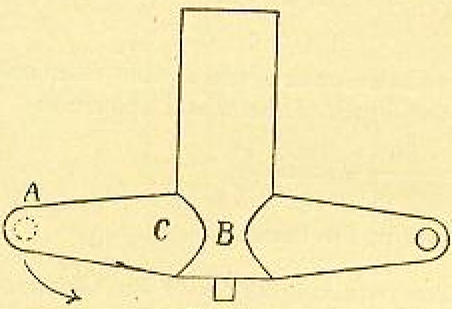

FIG. 15.

Since the water issues perpendicularly to the arms $\gamma_2 = 0$; and since the initial elements of the arms are radial, $\gamma_1 = 90°$, and as the water must flow radially into the arms, $\alpha = 90°$. The inner radius is necessarily small and may be considered zero. Hence, making

$$\gamma_2 = 0,\ \gamma_1 = 90°,\ \alpha = 90°,\ r_1 = 0,$$

in equation (14) gives

$$E = \frac{U}{\delta QH} = \frac{\omega r_2}{gH}\left[-\omega r_2 + \sqrt{\frac{2gH + \omega^2 r_2^2}{1 + \mu_2}}\right]. \quad . \quad (105)$$

Equation (19) gives

$$v_2 = \sqrt{\frac{2gH + \omega^2 r_2^2}{1 + \mu_2}}; \quad . \quad . \quad . \quad . \quad . \quad (106)$$

hence, the efficiency reduces to, for the frictionless wheel,

$$E = \frac{2\omega r_2}{v_2 + \omega r_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

This has no algebraic maximum, but approaches unity as the velocity increases indefinitely. Practically it has been found that the best effect is produced when the velocity of the orifices is about that due to the head, or

$$\omega r_2 = \sqrt{2gH}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

for which value the efficiency will be, if $\mu_2 = 0.10$

$$E = 2\left[-1 + \sqrt{\frac{2}{1.1}}\right] = 0.70. \quad . \quad . \quad . \quad . \quad (109)$$

If k_2 be the area of the effective section of the orifice, then

$$Q = k_2 v_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (110)$$

The pressure on the back side the arms opposite the orifices and useful in driving the mill, will be

$$P_1 = Mv_2 = \frac{\delta Q}{g} v_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (111)$$

Of this pressure there will be required

$$P_2 = M.\omega r_2 = \frac{\delta Q}{g}.\omega r_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

to impart to the water the rotary velocity ωr_2 which it has when it reaches the orifice. The effective pressure will be $P_1 - P_2$, and the work done per second will be this pressure into the distance it traverses per second, or

$$U = \omega r_2 [P_1 - P_2],$$

which reduces to the value found from equations (105) and (106).

36. *Exercise.*—Let the supply chamber be square, and from two of its opposite sides let pyramidal arms project. Let

$H = 10$ feet, orifices each 2 square inches, vertical section of arms through the orifice each 4 square inches, section of the arms where they join the supply chamber each 8 square inches, horizontal section of the supply chamber 36 square inches, $r_2 = 36$ inches, velocity of the orifice $\omega r_2 = \sqrt{2gH}$, coefficient of discharge 0.64, and $\mu_2 = 0.10$.

Required:

Velocity of discharge relative to the orifice,	$v_2 =$
Velocity of discharge relative to the earth,	$w =$
Velocity at entrance to the arms, . . .	$v_1 =$
Velocity in the supply chamber, . . .	$=$
The volume of water discharged, . . .	$Q =$
The weight of water discharged, . . .	$\delta Q =$
The work per second, ...	$U =$
The horse-power, ...	$HP =$
The efficiency, ...	$E =$
The pressure on arm opposite orifice at A per square inch, ...	$p_1 =$
The pressure at base of the arms at C, .	$p =$
The equation to the path of the fluid.	

37. *Scottish and Whitelaw Turbines.*—These wheels have no guide plates, and differ from Barker's mill chiefly in having curved arms. The analysis is precisely the same as for the Barker's mill. The only practical difference consists in providing a curved path for the water, instead of compelling the water to seek its path, forming eddies, etc.

38. *Jet Propeller.*—We first show how this problem may be solved by the preceding equations, and afterwards make an independent solution. Let a narrow vessel, Fig. 16, be carried by an arm E about a shaft BA. Let water, by any suitable device, be dropped into the vessel, the horizontal velocity of the water being the same as that of the vessel. At F, the lower end of this chamber, let there be an orifice from which water may issue horizontally. The water may then be con-

sidered as entering the vessel or bucket without velocity, and passing downward finally curve towards, and issue from, the orifice. It thus becomes a parallel flow wheel without guides, and we have, for the frictionless wheel,

$$r_1 = r_2, \quad \gamma_1 = 90^\circ, \quad \gamma_2 = 0, \quad \mu_1 = \mu_2 = 0, \quad H = 0,$$
$$p_2 = p_1 = p_a, \quad z_1 = 0$$

in equation (8); hence, the velocity of exit relative to the orifice will be

$$v_2^2 = 2gz_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (113)$$

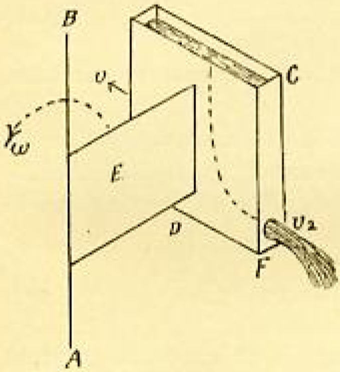

FIG. 16.

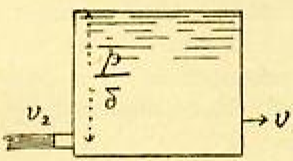

FIG. 17.

where z_2 equals the head in the supply chamber. Under these conditions the velocity of discharge will be independent of the velocity of rotation, if the rotation be uniform.

Equation (11) gives for the velocity of discharge relative to the earth

$$w = v_2 - \omega r_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (114)$$

Equations (19) and (14) give

$$U_1 = \frac{\delta Q}{g} v_2 \cdot r_2 \omega. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (115)$$

This equation may be factored thus, $\frac{\delta Q}{g}$ is the mass of liquid flowing out per second; represent by M; Mv_2 is the momentum of the outflowing liquid per second. Mv_2r_2 is the moment of the momentum, and, finally,

$Mv_2 \cdot r_2\omega$ is the moment of the momentum into the angular velocity, and equals the work done.

Let $v = \omega r_2 =$ the velocity of the vessel; then from (114) and (115)

$$w = v_2 - v, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (116)$$

$$U_1 = Mv_2v; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (117)$$

which equations are true whether the motion be circular, linear, or in any other path.

In practice, the velocity of the jet is produced by the pressure exerted by a pump, in which case z_2 in equation (113) would be replaced by a virtual head, Fig. 17, equivalent to z_2; or

$$v_2^2 = 2g\frac{p}{\delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (118)$$

Also the vessel, instead of having water supplied to it at the velocity of the vessel, picks it up from a body of water considered at rest; thus imparting to the water the momentum Mv, requiring the work per second

$$U_2 = Mv^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (119)$$

Hence the effective work done by a jet propeller picking up the water from a state of rest will be

$$U = U_1 - U_2 = M(v_2 - v)v. \quad . \quad . \quad . \quad . \quad (120)$$

The energy exerted by the pump will be that producing the velocity of water relative to the earth, or $\frac{1}{2}M(v_2 - v)^2$, plus that doing the work of driving the vessel; hence, the energy expended will be

$$\tfrac{1}{2} M(v_2 - v)^2 + U;$$

and the efficiency will be

$$E = \frac{U}{U + \tfrac{1}{2} M(v_2 - v)^2} = \frac{2v}{v_2 + v}. \quad . \quad . \quad . \quad (121)$$

This has no algebraic maximum, but approaches unity as v, the velocity of the vessel, in reference to the earth, approaches v_2 in value, the velocity of the jet in the opposite direction relative to the orifice.

If $v_2 = v$, *the efficiency will be perfect* as shown by (120), *but no work will be done* as shown by (119). This would be the case of a vessel drawn by an external agency, or even floating along a stream; for the water backward relative to the vessel would equal the forward velocity of the vessel.

The mass of the jet per second will vary as the section of the orifice and velocity of the jet; and if k be the section of the jet, then

$$U = \frac{\delta}{g} k v_2 (v_2 - v) v; \quad . \quad . \quad . \quad . \quad . \quad (122)$$

hence the same work may be done by enlarging the section k, and properly diminishing the velocity v_2 of the jet; but as v_2 is diminished, the efficiency is increased, as shown by equation (121).

If $v = 10$ feet per second (about 6.8 miles per hour), we find

v_2	U	k	E
10	$0. k \frac{\delta}{g}$	∞	1.00
15	750 k "	120.0	0.80
20	2000 k "	45.0	0.67
30	6000 k "	15.0	0.50
40	12000 k "	7.5	0.40
100	90000 k "	1.0	0.16

The sections k here given are for equal works, U. If the velocity of exit be constant, then will the work increase directly as the area of the section while the efficiency remains the same. These are without frictional resistances.

The pressure against the side of the vessel opposite the orifice due to the reaction of the water will be found from equation (117) by dividing the work done by the space over which the work is done, or

$$P_1 = \frac{U_1}{v} = M v_2, \quad . \quad . \quad . \quad . \quad . \quad (123)$$

which is *the momentum of the jet per second relative to the orifice.*

To impart to the water taken up the uniform velocity v would require the constant pressure

$$P_2 = Mv;$$

hence the resultant pressure producing work would be

$$P = P_1 - P_2 = M(v_2 - v), \quad . \quad . \quad . \quad . \quad (124)$$

and the resultant work would be

$$U_2 = M(v_2 - v)v, \quad . \quad . \quad . \quad . \quad . \quad . \quad (125)$$

as before found in equation (120).

The speed of a jet propeller depends upon the form of the vessel and the nature of the fluid; but the pressures due to the action of the jet will be the same whether it issue into a vacuum, or into air or water, or a more viscous fluid. If a block be placed before the jet so close to the vessel as to obstruct the flow of water as a jet, the conditions will be changed, and the forward pressure will then be due partly to the direct pressure exerted by the pumps. If a piston, having a long piston-rod projecting against a firm body outside the vessel, be forced backward, the forward pressure, effective in driving the vessel, would be that exerted by the pumps less the frictional resistances.

The efficiency of the jet propeller as a motor is comparatively small in practice. This is due to the great loss of energy in the jet. The *entire* energy in the jet is lost. If the vessel be anchored, and the velocity of the jet be v_2, the pressure will be

$$P_1 = Mv_2,$$

the work will be

$$U = 0,$$

and the

$$\textit{energy lost} = \tfrac{1}{2}Mv_2^2.$$

If the speed v of the vessel is small, then

$$P_1 = Mv_2,$$

$$U = Mv_2v, \text{ nearly},$$

$$\textit{energy lost} = \tfrac{1}{2}Mv^2_2, \text{ nearly},$$

and the energy lost will generally exceed considerably the useful work.

39. *The difference of the moment of the momentum of the water on entering and leaving the wheel, equals the moment exerted by it on the wheel.* (Proof for the frictionless turbine by J. Lester Woodbridge, graduate of Stevens Institute, 1886.)

Consider the effect of water passing along a smooth, curved horizontal tube rotating about a vertical axis. Conceive the water to be divided into an infinite number of filaments by

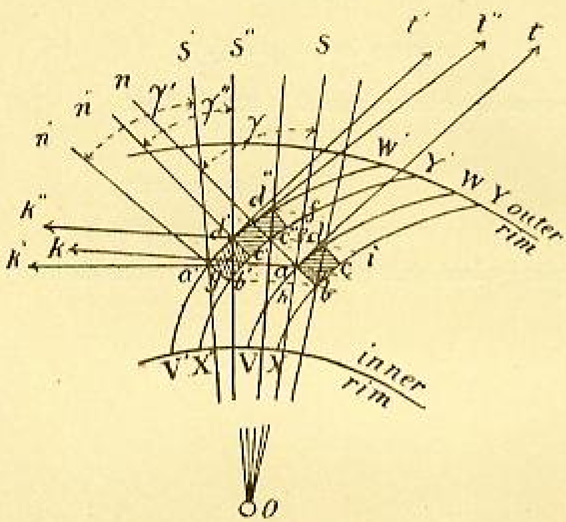

FIG. 18.

vanes similar to those of the wheel, but subjected to the condition that, at each point, their width, *id*, Fig. 18, measured on the arc, whose centre is *O*, shall subtend at the centre a constant angle $d\theta$. Conceive each filament to be divided into small prisms, whose bases are represented by the shaded areas $a'b'c'd'$, $d'c'c''d''$ and $abcd$, by vertical planes normal to the vanes making the divisions ae, ef, intercepted on the radius by circles passing through the consecutive vertices on the same vane, a', d', d'', etc., equal.

Let ρ = the radius vector ;

x = the height of an elementary prism ;

then, $d\rho = ae, ef$, etc. ;

$\rho d\theta d\rho = abcd$, etc., = area of the base of an infinitesimal prism ;

$x\rho d\theta d\rho$ = volume of an infinitesimal prism ;

$x\delta\rho d\theta d\rho = m$ = the mass of prism, δ being its density or mass of unit volume ;

$\gamma = san$ = angle between the normal to the vane at any point, and the radius Oa prolonged through that point ;

v = velocity of a particle along the vane at ρ, which is assumed to be the same in all the vanes at the same distance from the centre ;

ω = the uniform angular velocity of the wheel, and

p = the pressure of the water at the point ρ due to a head, but not due to deflection.

Let ρ be the independent variable, and dt the time required for the element $a'b'c'd'$ to move its own length, $d\rho$, and aa' the distance passed through by this element circumferentially in the same time, dt, then

$$dt = \frac{d\rho}{v \sin \gamma},$$

and,

$$aa' = \omega\rho dt = \omega\rho \frac{dt}{d\rho} d\rho.$$

The mass m will have two motions: one along the vane, the other with the wheel perpendicular to the radius. By changing its position successively in each of these directions, both its velocity with the wheel and its velocity along the vane may suffer changes both in *amount* and *direction*, as follows:

(I.) By moving from a to a', Fig. 18, in the arc of a circle—

(1.) $\omega\rho$ may be increased or diminished ;

(2.) $\omega\rho$ may be changed in direction ;

(3.) v may be increased or diminished;
(4.) v may be changed in direction.

(II.) By moving from a' to d' along the vane—
(5.) $\omega\rho$ may be increased or diminished;
(6.) $\omega\rho$ may be changed in direction;
(7.) v may be increased or diminished;
(8.) v may be changed in direction.

These changes give rise to corresponding reactions, as follows:

(No. 1.) Since the element is to move from a to a' in the arc of a circle, $\omega\rho$ will be constant, and hence the reaction $= 0$.

(No. 2.) By moving from a to a', the velocity $\omega\rho$ is changed in direction from ak to $a'k'$ in the time dt. The momentum is $m\omega\rho$, and the *rate* of angular change is

$$\frac{kak'}{dt} = \frac{\omega\, dt}{dt} = \omega,$$

and hence the force upon the element producing motion in the arc of a circle will be radially inward and the reaction will be $m\omega^2\rho$ radially outward. This is generally called the *centrifugal force*, as designated by most writers. Resolving into two components, we have

$$m\omega^2\rho \sin\gamma \text{ along the vane,}$$
$$m\omega^2\rho \cos\gamma \text{ normal to the vane.}$$

(No. 3.) According to the conditions imposed, this value of v is the same at a' as at a, hence, for this case, the reaction will be zero.

(No. 4.) In moving from a to a' the velocity along the vane, v, is changed in direction from at to $a't'$ at the rate ω as in No. 2.

The momentum is mv, and the force will be $mv\omega$, which acts in the direction bn. Since the particle will be driven by the vane XY, and the reaction will be in the direction nb; which being resolved, gives

$$0 \text{ along the vane,}$$
$$-mv\omega \text{ normal to the vane.}$$

(No. 5.) In passing from a' to d'; at d' the circular velocity will be greater than at a' by the amount

$$\omega d\rho,$$

and the acceleration will be

$$\omega \frac{d\rho}{dt},$$

requiring a force $m\omega \frac{d\rho}{dt}$ tangentially to the wheel in the direction of motion, the reaction of which will be

$$m\omega \frac{d\rho}{dt},$$

but backwards, and its components will be

$$m\omega \frac{d\rho}{dt} \cos \gamma \text{ along the vane,}$$

$$-m\omega \frac{d\rho}{dt} \sin \gamma \text{ normal to the vane.}$$

(No. 6.) In passing from a' to d', $\omega\rho$ will be changed in direction by the angle between $k'a'$ and $k''d'$, or

$$a'Od' = \frac{a'g}{\rho} = \frac{d\rho \cot \gamma}{\rho},$$

and the rate of angular change will be

$$\frac{\cot \gamma}{\rho} \cdot \frac{d\rho}{dt},$$

and the momentum being

$$m\omega\rho,$$

the reaction will be

$$m\omega \cot \gamma \frac{d\rho}{dt},$$

which acts radially inward and its components are

$$-m\omega \cos \gamma \frac{d\rho}{dt} \qquad \text{along the vane,}$$

$$-m\omega \cot \gamma \cos \gamma \frac{d\rho}{dt} \text{ normal to the vane.}$$

(No. 7.) By moving from a' to d', v will be increased by an amount

$$\frac{dv}{d\rho} d\rho,$$

in the time dt, and the reaction will be

$$m \frac{dv}{d\rho} \cdot \frac{d\rho}{dt}$$

which will be outward along the vane, and the reaction will be directly backward along the vane, and hence is

$$-m \frac{dv}{d\rho} \cdot \frac{d\rho}{dt} \quad \text{along the vane,}$$

$$0 \quad \text{normal to the vane.}$$

(No. 8.) In passing from a' to d', v is changed in direction by two amounts: the angle γ changes an amount

$$d(\gamma'' - \gamma') = -\frac{d\gamma}{d\rho} d\rho.$$

This is negative, for a differential is the limiting value of the second state minus the first, and the first is here larger.

But this is not the total change, since γ'' is measured from a radius making an angle

$$\frac{d\rho \cot \gamma}{\rho},$$

with Oa' as in No. 6; hence the total change will be the sum of these, and the *rate* of change will be the sum divided by dt, which result, multiplied by the momentum mv, will give the reaction, which will be normal and in the direction $b'n'$ or

$$0 \qquad \text{along the vane,}$$

$$mv\left[\frac{\cot \gamma}{\rho} \cdot \frac{d\rho}{dt} - \frac{d\gamma}{d\rho} \cdot \frac{d\rho}{dt}\right], \text{ normal to the vane.}$$

This completes the reactions. Next consider the *pressure in the wheel*. The *intensity* of the pressure on the two sides ab and cd differs by an amount

$$dp = \frac{dp}{d\rho} d\rho.$$

The area of the face is $dc \times x = x\rho d\theta \sin \gamma$, and the force due to the difference of pressures will be

$$x\rho d\theta \sin \gamma \frac{dp}{d\rho} d\rho.$$

If dp is positive, which will be the case when the pressure on dc exceeds that on ab, the force acts backwards, and the preceding expression will be *minus* along the vane. In regard to the pressure normal to the vane, if a uniform pressure p existed from one end of the vane VW to the other, the resultant effect would be zero, since the pressure in one direction on VW would equal the opposite pressure on XY. If, however, in passing from d to e, the pressure increases by an amount $-dp$, since Va is longer than Xb, the pressure on Va will exceed that on Xb by an amount

$$-dp \, . \, x \times ah = -dp \, . \, x \, . \, \rho d\theta \cos \gamma = -x\rho \cos \gamma d\theta \frac{dp}{d\rho} d\rho.$$

Collecting these several reactions, we have

	NORMAL TO THE VANE.	ALONG THE VANE.
(1.)	0	0
(2.)	$+ m\omega^2 \rho \cos \gamma.$	$+ m\omega^2 \rho \sin \gamma.$
(4.)	$- m\omega v.$	$0.$
(5.)	$- m\omega \sin \gamma \frac{d\rho}{dt}.$	$+ m\omega \cos \gamma \frac{d\rho}{dt}.$
(6.)	$- m\omega \cot \gamma \cos \gamma \frac{d\rho}{dt}.$	$- m\omega \cos \gamma \frac{d\rho}{dt}.$
(7.)	0	$- m \frac{d\rho}{dt} \cdot \frac{dv}{d\rho}.$
(8.)	$+ mv \left[\frac{\cot \gamma}{\rho} \cdot \frac{d\rho}{dt} - \frac{d\gamma}{d\rho} \cdot \frac{d\rho}{dt} \right].$	$0.$
(9.)	$- x\rho \cos \gamma \frac{dp}{d\rho} d\rho d\theta.$	$- x\rho \sin \gamma \frac{dp}{d\rho} d\rho d\theta$

The sum of the quantities in the second column, neglecting friction, will be zero; hence

$$m\omega^2 \rho \sin \gamma - m \frac{d\rho}{dt} \cdot \frac{dv}{d\rho} - x\rho \sin \gamma \frac{dp}{d\rho} d\rho d\theta = 0. \quad . \quad (123)$$

Substituting

$$\frac{d\rho}{dt} = v \sin \gamma, \text{ and } x\rho d\theta d\rho = \frac{m}{\delta}$$

and dividing by $m \sin \gamma$, we have

$$\omega^2 \rho d\rho - \frac{1}{\delta} dp = v dv. \quad . \quad . \quad . \quad . \quad . \quad (124)$$

Integrating,

$$\left[\tfrac{1}{2}\omega^2\rho^2 - \frac{p}{\delta}\right]_{\text{limit}}^{\text{limit}} = \left[\tfrac{1}{2}v^2\right]_{\text{limit}}^{\text{limit}} . \quad . \quad . \quad . \quad (125)$$

The sum of the quantities in the first column gives the pressure normal to the vane, which, multiplied by $\rho \sin \gamma$, gives the moment. This done, we have

$$d^2M = mv \sin \rho \left\{ \begin{array}{l} \omega\gamma \left(\frac{\rho}{v}\,\omega \cos \gamma - 2\right) - \rho v \sin \gamma \frac{d\gamma}{d\rho} \\ + v \cos \gamma - \rho \frac{\cos \gamma}{v\delta}\frac{dp}{d\rho} \end{array} \right\}$$

Putting $mv \sin \gamma = \frac{\delta Q}{2\pi} d\rho d\theta$, where Q is the quantity of water flowing through the wheel per second, and integrating in reference to θ between 0 and 2π, we have

$$dM = \delta Q \left[\omega\rho\left(\frac{\rho}{v}\omega \cos \gamma - 2\right) - \rho v \sin \gamma \frac{d\gamma}{d\rho} + v \cos \gamma - \rho \frac{\cos \gamma}{v\delta}\frac{dp}{d\rho}\right] d\rho$$

Multiplying (124) by

$$\frac{\rho}{v} \cos \gamma,$$

we have

$$\frac{\omega^2\rho^2}{v} \cos \gamma d\rho - \frac{\rho \cos \gamma}{v\delta}\frac{dp}{d\rho} d\rho = \rho \cos \gamma \frac{dv}{d\rho} d\rho$$

which substituted above gives

$$dM = \delta Q \left[-2\omega\rho d\rho + \rho \cos \gamma \frac{dv}{d\rho} d\rho + v \cos \gamma d\rho - \rho v \sin \gamma \frac{d\gamma}{d\rho} d\rho \right] (126)$$

the integral of which is

$$\begin{aligned} M &= \delta Q \left[-\omega\rho^2 + \rho v \cos \gamma \right] \\ &= -\delta Q \rho \left[\omega\rho - v \cos \gamma \right]_{\text{limit.}}^{\text{limit.}} . \quad . \quad . \quad (127) \end{aligned}$$

But $\omega \rho - v \cos \gamma$ is the circumferential velocity in space of the water at any point, and $\delta Q \rho [\omega \rho - v \cos \gamma]$ is the moment of the momentum; hence, integrating between limits for inner and outer rims, *the moment exerted by the water on the wheel equals the difference in its moment of momentum on entering and leaving wheel.*

Let the values of the variables at the entrance of the wheel be ρ_1, γ_1, v_1, p_1, and at exit ρ_2, γ_2, v_2, p_2.

Equations (125) and (127) become

$$\tfrac{1}{2} \omega^2 (\rho_1^2 - \rho_2^2) - \frac{p_1 - p_2}{\delta} = \tfrac{1}{2} (v_1^2 - v_2^2). \quad . \quad . \quad (128)$$

$$M = \delta Q [\omega (\rho_1^2 - \rho_2^2) - \rho_1 v_1 \cos \gamma_1 + \rho_2 v_2 \cos \gamma_2]. \quad (129)$$

$$U = M \omega = \delta Q \omega [\omega (\rho_1^2 - \rho_2^2) - \rho_1 v_1 \cos \gamma_1 + \rho_2 v_2 \cos \gamma_2, \quad (130)$$

which is essentially the same as equation (58). It, however, involves the velocity of entrance, v_1, and of exit, v_2. The former may be found by equation (6), when α is known, or assumed when α is to be determined, and v_2 may be found by (19) or (56). This principle does not appear to be of great value in the solution of the general problem, but may be of much service in certain special cases. Thus, in the Barker Mill, page 44, the moment of the momentum of the water entering the wheel will be zero, but of exit will be

$$M V,$$

where V is the velocity of exit, relative to the earth, perpendicular to the arm, and the moment will be

$$\begin{aligned} &M V r_2; \\ \therefore\ U &= M V \cdot r_2 \omega \\ &= Ps, \qquad (131) \end{aligned}$$

where P is the pressure on the arm opposite the orifice, and s the space passed over by the orifice in a second of time.

But

$$V = v_2 - r_2\,\omega;$$

$$M = \frac{\delta\,Q}{g} = \frac{\delta\,k_2\,v_2}{g};$$

$$(1 + \mu)\,v_2 = 2\,g\,H + \omega^2\,r_2^2;$$

$$\therefore\ U = \frac{\delta\,k_2\,v_2}{g}\,(v_2 - r_2\,\omega)\,r_2\,\omega \qquad (132)$$

$$= \frac{\delta\,Q}{g}\left(-\,r_2\,\omega + \sqrt{\frac{2\,g\,H + \omega^2\,r_2^2}{1 + \mu}}\right)r_2\,\omega,$$

as in equation (105).

40. Again, if the water quits the wheel radially, then the moment of the momentum of the quitting water will be zero, and

$$U = M\,V\,r_1\,cos\,\alpha \cdot \omega.$$

But

$$V\,cos\,\alpha = V_1,$$

the tangential component of the velocity, or velocity of whirl;

$$\therefore\ U = M\,V_1\,r_1\,\omega. \qquad (133)$$

41. *In the frictionless Rankine wheel* the velocity of whirl equals the velocity of the initial rim of the wheel.

$$\therefore\ V_1 = r_1\,\omega;$$

$$\therefore\ U = M\,r_1^2\,\omega^2. \qquad (134)$$

The work will also equal the potential energy of the water, $W\,H = \delta\,Q\,H$, less the kinetic energy of the quitting water, $\frac{1}{2}\,M\,w_1^2$ (less the energy lost in resistances, $\mu\,v_2^2$, which in this case we neglect);

$$\therefore\ U = \delta\,Q\,H - \tfrac{1}{2}\,M\,w^2,$$

and since the water is assumed to quit radially

$$w = r_2\,\omega \cdot tan\,\gamma_2 = r_1\,\omega\,tan\,\alpha. \qquad (135)$$

The three preceding equations give

$$r_1\,\omega = \sqrt{\frac{2\,g\,H}{2 + tan^2\,\alpha}}, \tag{135}$$

as in equation (76).

42. *Again, if the crowns are parallel discs* and the initial element of the bucket is radial, and if the water quits the wheel radially, and if the velocity of whirl equals the velocity of the initial rim, we have

$$U = M\,r_1^2\,\omega^2, \tag{136}$$

as in equation (134). But γ_2 will not be the same as in (135). To find it we have, neglecting the thickness of the walls of the buckets,

$$2\,\pi\,r_1\,v_1 = 2\,\pi\,r_2\,sin\,\gamma_2 \cdot v_2$$
$$v_1 = V\,sin\,\alpha$$
$$r_1\,\omega = V\,cos\,\alpha$$
$$w = r_2\,\omega\,tan\,\gamma_2$$
$$v_2 = r_2\,\omega\,;$$
$$\therefore\ tan\,\gamma_2 = \frac{r_1^2\,tan\,\alpha}{\sqrt{r_2^4 - r_1^4\,tan^2\,\alpha}}\,; \tag{137}$$

$$\therefore U = \delta\,Q\,H - \tfrac{1}{2}\,M\,w^2$$
$$= \delta\,Q\,H - \tfrac{1}{2}\,M\,r_2^2\,\omega^2\,\frac{r_1^4\,tan^2\,\alpha}{r_2^4 - r_1^4\,tan^2\,\alpha}. \tag{138}$$

Equations (136) and (138) give

$$r_1\,\omega = \sqrt{\frac{2\,g\,H}{2 + \dfrac{r_1^2\,r_2^2\,tan^2\,\alpha}{r_2^4 - r_1^4\,tan^2\,\alpha}}}. \tag{139}$$

43. *Conclusions.*—From an examination of Tables II., III., VI., VII., the following conclusions are drawn:

1. The maximum theoretical efficiency of the inflow wheel is perceptibly larger than that of the outflow, the width of crowns and the initial and terminal angles of the buckets being the same. One reason for this is due to the flow through the wheel being opposed by the centrifugal action, but more particularly to the smaller velocity of discharge from the inflow wheel.

2. Columns (10) in Tables VI. and VII. show that for the wheels here considered the loss of energy due to the quitting velocity is from 2.2, 5.1 per cent. from the outflow, and from 0.9 to 1 per cent for the inflow.

3. The same tables show that in column (2) the efficiency is almost constant for the varying conditions here considered, while for the outflow there is considerable variation.

4. One of the most interesting and profitable studies to the theorist and practitioner is the effect upon the efficiency due to properly proportioning the terminal angle, α, of the guide blade. It will be observed that all the efficiencies in Tables VI. and VII. exceed the corresponding ones in Table II. except the first in column (3) of Table II. In Table II. the terminal angle, α, is constantly 25°, while in Tables VI. and VII. it is less than that value, and in the highest efficiencies very much less.

5. It appears from these tables that the terminal angle, α, has frequently been made too large for best efficiency.

6. That the terminal angle, α, of the guide should be comparatively small for best effect; for the inflow less than 10°, and that theoretically, when the angle is about 7°, the efficiency is some 10 per cent. greater than when it is 25° in the wheels here considered.

7. Tables II. and VI. indicate that the initial angle of the bucket should exceed 90° for best effect for outflow wheels.

8. Tables II. and VII. show that the initial angle should be less than 90° for best effect on inflow wheels, but that from 60° to 120° the efficiency varies scarcely 1 per cent.

9. The most marked effect in properly proportioning the terminal angle, α, of the guide is shown when the initial angle of the bucket is 150°. In this case the efficiency for the outflow when α is 25° is 0.744, Table II., but when α is $13\frac{1}{2}$°, as in Table VI., it becomes 0.921. For the inflow, in the former case, it is 0.752, but when the angle is 3°, as in Table VII., it becomes 0.918.

10. Since the wheels here considered have the same width of crowns and the same terminal angle of the bucket, the depths of the wheels will be proportional to k_2 for discharging equal volumes of water. Tables III., VI., VII. show that the section k_2 increases as the initial angle of the buckets increases, and that it must be greater for the inflow than for the outflow; hence the depth of the wheel must be greater for the inflow for delivering the same volume of water.

11. But the same volume of water delivered by the inflow does more work than that of the outflow; the depths should be as k_2, divided by the efficiency. Thus in Tables VI. and VII., for $\gamma = 90°$, and for the same heads, H, the relative depths should be for equal works $(0.759 \div 0.828) \div (150 \div 0.920) = 1.67$.

12. In the outflow wheel, column (9), Table VI., shows that for the outflow for best effect the direction of the quitting water in reference to the earth should be nearly radial (from 76° to 97°), but for the inflow wheel the water is thrown forward in quitting (column [9] Table VII.). This alone shows that the velocity of the rim should somewhat exceed the relative final velocity backward in the bucket, as shown in columns (4) and (5).

13. In these tables I have given all the velocities in terms of $\sqrt{2gh}$, and the coefficients of this expression will be the part of the head which would produce that velocity if the water issued freely. In Tables VI. and VII. there is only one case, column (5) of the former table, where the coefficient exceeds unity, and the excess is so small it may be discarded; and it may be said that in a properly proportioned turbine with the conditions here

given, none of the velocities will equal that due to the head in the supply chamber when running at best effect.

14. The inflow turbine presents the best conditions for construction for producing a given effect, the only apparent disadvantage being an increased first cost due to an increased depth, or an increased diameter for producing a given amount of work. The larger efficiency should, however, more than neutralize the increased first cost.

Column (3) shows that the efficiency, E, increases as the initial angle of the bucket, γ_1, increases up to 120°. The maximum will be for about 120° with this amount of friction.

TABLE VI.

OUTWARD FLOW TURBINE. $r_1 = r_2 \sqrt{\frac{1}{2}}$ $\epsilon_1 = \epsilon_2 = 0.10$. $\gamma_2 = 12°$. Parallel crowns. $k_1 v_1 = k_2 v_2 = KV = Q = 1$.

Initial angle. γ_1	Efficiency E. Eq. (61).	Velocity outer rim. $r_2\omega'$ Eq. (60).	Velocity inner rim. $r_1\omega' = \sqrt{\frac{1}{2}}\, r_2\omega'$	Relative velocity of exit. v_2 Eq. (56).	Relative velocity of entrance. v_1 Eq. (57).	Velocity of exit from supply chamber. V Eq. (55).	Terminal angle of guide. α Eq. (62).	Direction of quitting water θ Eq. (35).	Head equivalent of energy in quitting water. $\frac{w^2}{2g}$	$k_2 \sqrt{gH}$ Eqs. (20); (56).	$k_1 \sqrt{gH}$ Eqs. (20); (57).	$K \sqrt{gH}$ Eqs. (20); (55).
1	2	3	4	5	6	7	8	9	10	11	12	13
60°	0.804	$1.374 \sqrt{gH}$ $0.972 \sqrt{2gH}$	$0.972 \sqrt{gH}$ $0.687 \sqrt{2gH}$	$1.483 \sqrt{gH}$ $1.048 \sqrt{2gH}$	$0.503 \sqrt{gH}$ $0.356 \sqrt{2gH}$	$0.842 \sqrt{gH}$ $0.595 \sqrt{2gH}$	31° 17′	76°	$0.051H$	0.67	1.988	1.187
90°	0.828	$1.236 \sqrt{gH}$ $0.874 \sqrt{2gH}$	$0.874 \sqrt{gH}$ $0.619 \sqrt{2gH}$	$1.317 \sqrt{gH}$ $0.931 \sqrt{2gH}$	$0.387 \sqrt{gH}$ $0.274 \sqrt{2gH}$	$0.956 \sqrt{gH}$ $0.676 \sqrt{2gH}$	25° 56′	79°	$0.039H$	0.76	2.584	1.046
120°	0.841	$1.129 \sqrt{gH}$ $0.796 \sqrt{2gH}$	$0.798 \sqrt{gH}$ $0.563 \sqrt{2gH}$	$1.192 \sqrt{gH}$ $0.843 \sqrt{2gH}$	$0.405 \sqrt{gH}$ $0.367 \sqrt{2gH}$	$1.060 \sqrt{gH}$ $0.819 \sqrt{2gH}$	19° 5′	82°	$0.031H$	0.84	2.444	0.943
150°	0.921	$1.005 \sqrt{gH}$ $0.709 \sqrt{2gH}$	$0.709 \sqrt{gH}$ $0.501 \sqrt{2gH}$	$0.999 \sqrt{gH}$ $0.707 \sqrt{2gH}$	$0.587 \sqrt{gH}$ $0.416 \sqrt{2gH}$	$1.252 \sqrt{gH}$ $0.886 \sqrt{2gH}$	13° 31′	97°	$0.022H$	1.00	1.703	0.799

TABLE VII.

INWARD FLOW TURBINE. $r_1 = \sqrt{2}\, r_2$ $\epsilon_1 = \epsilon_2 = 0.10$ $\gamma_2 = 12°$. Parallel crowns. $k_1 v_1 = k_2 v_2 = KV = Q = 1$.

γ_1	E. Eq. (61).	Velocity outer rim. $r_1\omega'$	Velocity inner rim. $r_2\omega'$	v_2 Eq. (56).	v_1 Eq. (57).	V Eq. (55).	α Eq. (62).	θ Eq. (35).	$\frac{w^2}{2g}$	$k_2 \sqrt{gH}$ Eqs. (20); (56).	$k_1 \sqrt{gH}$ Eqs. (20); (57).	$K \sqrt{gH}$ Eqs. (20); (55).
1	2	3	4	5	6	7	8	9	10	11	12	13
60°	0.920	$1.003 \sqrt{gH}$ $0.709 \sqrt{2gH}$	$0.709 \sqrt{gH}$ $0.501 \sqrt{2gH}$	$0.673 \sqrt{gH}$ $0.476 \sqrt{2gH}$	$0.115 \sqrt{gH}$ $0.089 \sqrt{2gH}$	$0.951 \sqrt{gH}$ $0.672 \sqrt{2gH}$	7° 0′	110°	$0.010H$	1.48	8.65	1.05
90°	0.930	$0.973 \sqrt{gH}$ $0.688 \sqrt{2gH}$	$0.688 \sqrt{gH}$ $0.487 \sqrt{2gH}$	$0.664 \sqrt{gH}$ $0.470 \sqrt{2gH}$	$0.098 \sqrt{gH}$ $0.069 \sqrt{2gH}$	$0.977 \sqrt{gH}$ $0.691 \sqrt{2gH}$	5° 28′	106°	$0.010H$	1.50	10.20	1.02
120°	0.919	$0.945 \sqrt{gH}$ $0.668 \sqrt{2gH}$	$0.668 \sqrt{gH}$ $0.473 \sqrt{2gH}$	$0.645 \sqrt{gH}$ $0.456 \sqrt{2gH}$	$0.109 \sqrt{gH}$ $0.077 \sqrt{2gH}$	$1.004 \sqrt{gH}$ $0.709 \sqrt{2gH}$	4° 46′	105°	$1.010H$	1.55	9.17	0.99
150°	0.918	$0.897 \sqrt{gH}$ $0.634 \sqrt{2gH}$	$0.634 \sqrt{gH}$ $0.448 \sqrt{2gH}$	$0.607 \sqrt{gH}$ $0.429 \sqrt{2gH}$	$0.178 \sqrt{gH}$ $0.126 \sqrt{2gH}$	$1.053 \sqrt{gH}$ $0.743 \sqrt{2gH}$	3° 08′	107°	$0.009H$	1.65	5.05	0.95