Lorenzo ALLIEVI

THEORY
OF
WATER-HAMMER

Translated by Mr. Eugène E. HALMOS

NOTES I TO V

(TEXT)

ROME
TYPOGRAPHY RICCARDO GARRONI
Piazza Mignanelli, 23
1925

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W P Creager

With the compliments
of the translator
Eugene E. Halmos

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NOTE I.

GENERAL DISCUSSION OF THE METHOD

§. 1. — General comments on the variable flow.

In a preceding paper, (*) the notations of which will be maintained in general, I have given the general formulas which govern the perturbed (variable) motion of water in pressure conduits.

I have also demonstrated that the pressure variations are propagated along the tube with a velocity a which is a function of the moduli of elasticity, E, ε, of the pipe and the liquid and also of the diameter D and the thickness e of a given conduit, according to the formula:

$$\frac{1}{a^2} = \frac{\omega}{g}\left(\frac{1}{\varepsilon} + \frac{1}{\mathrm{E}} \cdot \frac{\mathrm{D}}{e}\right) \qquad (1)$$

which for water in metal tubes, putting $\omega = 1000$ kg./m³, $\varepsilon = 2{,}07 \times 10^8$ kg./m² gives

$$a = \frac{1425}{\sqrt{1 + \frac{\varepsilon}{\mathrm{E}} \cdot \frac{\mathrm{D}}{e}}} \qquad (1\ bis)$$

Inserting the values E for steel and for cast iron, it was observed that the numerical value of a ranges from a minimum of 600 to 700 m. per sec., for thin pipes of large diameter, to a maximum of 1200 to 1300 m. per sec., for thick pipes of small diameter.

I have also demonstrated that the height of the variable pressure (expressed in meters of water) and the velocity v in any section of the conduit during the variable motion are expressed by the equations (**):

$$\left.\begin{aligned} y &= y_0 + \mathrm{F} + f \\ v &= v_0 - \frac{g}{a}(\mathrm{F} - f) \end{aligned}\right\} \qquad (2)$$

in which F and f signify the variable pressure heights expressed by functions

(*) See introduction.

(**) Contrary to the notation of the 1902 paper. I prefer not to make any limiting assumption as regards the sign of F and f and therefore have given to f the same sign as that of F in the first, and the opposite sign in the second equation of (2).

of the form

$$F\left(t - \frac{x}{a}\right) \text{ and } f\left(t + \frac{x}{a}\right) \text{ respectively,}$$

where x is reckoned in opposite direction to v; that is, F represents a variable pressure (positive or negative) being propagated in the direction $+x$ whith a velocity a, and conversely, f represents a variable pressure propagated in the direction $-x$ with the same velocity a. In fact F becomes a constant quantity if we put

$$x = + at + \text{const.}$$

(as it would appear to an observer traveling along the tube with a velocity a in the direction of $+x$) and conversely f becomes a constant quantity if we put

$$x = - at + \text{const.}$$

I have demonstrated, moreover, that equations (2), if the limiting conditions are introduced, can serve to determine the values F and f at all instants and at all sections of the tube, and that these equations make possible the numerical solution of the phenomena of variable motion in any given case. (*)

Considering a conduit of length L, having, at its lower end (the origin of the abcissa x), a gate which can vary the efflux, and communicating, at the upper end, whith a reservoir of constant level, it was demonstrated that, for each section of the pipe, the function f has at any given instant a value equal to and of opposite sign of that which the function F had at an instant which precedes the instant considered by a time interval

$$\frac{2\,(L - x)}{a},$$

which is the time interval necessary to travel twice, with a velocity a, the portion of the pipe between the section considered and the reservoir. We have therefore

$$f\left(t + \frac{x}{a}\right) = - F\left(t - \frac{x}{a} - \frac{2\,(L - x)}{a}\right) =$$
$$= - F\left(t + \frac{x}{a} - \frac{2L}{a}\right); \qquad (3)$$

and the phenomenon occurs as if every superpressure F propagated from the gate toward the reservoir (direction $+x$) were reflected with a negative sign (depression) and sent back toward the gate (direction $-x$).

(*) It can also be observed that if the law of the propagation of the variable pressures with a constant velocity a is admitted as an experimental fact, the 2nd equation of (2) can ben directly derived by applyng the general dynamical principles to the motion of a liquid element of thickness $dx = adt$.

It results evidently from the preceding, that in the case of such a conduit, the perturbances produced by the gate movement at any section of abcissa x will be perfectly known as soon as perturbances produced at the section of abcissa $x = o$, at the gate, are determined, and, where, by (3), we have

$$f(t) = -F\left(t - \frac{2L}{a}\right) \qquad (3\ bis)$$

Finally, I have demonstrated in my 1902 paper, that by the introduction of the efflux equation (which establishes another relation betwen the pressure height and the velocity of flow in the tube) it is easy to determine a series of numerical values of F for values of time differing by

$$\frac{2L}{a};$$

now, each of these values of F gives the values taken by f at the same abcissa, but at the succeeding instant, that is $\frac{2L}{a}$ seconds later, so that we can calculate the values of pressure and of velocity of the water for each of these instants.

In the following, I will designate by the symbol μ the interval $\frac{2L}{a}$ which I have called the duration of the phase. I have shown that if the operation of the gate is started when the regime of flow is permanent, the function f is constantly zero during the period of time equal to μ (duration of the direct blow) and that it is possible, moreover, by introducing the efflux equation, to calculate the value taken, at any instant t_1, between o and μ, by the first term F_1 of the series of values F_1, F_2, F_3, corresponding to the time

$$t_1, \quad t_1 + \mu, \quad t_1 + 2\mu, \text{ etc.};$$

each of the terms F_1, F_2, F_3, etc, gives, as already observed, the value which $-f$ will take μ seconds later.

The expression: duration of the phase, for the designation of the time interval $\frac{2L}{a}$ seems the more justified, as it is eesy to demonstrate that in the case of a waterhammer produced by the movement of a gate executed with constant speed, the law of variation of the pressure is subject to sudden discontinuities at the instants

$$t = \mu, \quad 2\mu, \quad 3\mu, \text{ etc.}$$

The graph of the pressure, as a function of time, will therefore show a broken line the angles of which are separated from each other by equal intervals corresponding to phases of duration μ.

I designated by « phases of counter-blow » the periods of time equal to μ which succeed the phase of the direct blow, and during which the phenomenon of the reflection of the pressure from the reservoir toward the gate (phenomenon of counter blow) gives values different from zero to the function f.

These general principles therefore suffice to solve numerically the phenomena of waterhammer for all given conditions, and in my previous paper, I have shown their application, to the most important practical cases, as for instance the closing and opening of the gate or the stopping of same at a given point, etc.

I have pointed out also, in certain cases, the influence of the relative magnitude of the constants defining the tube (pressure height y_0, normal velocity v_0, velocity of propagation a) and the speed of the gate operation, but these fragmentary investigations are far from constituting the « Theory of Waterhammer », and are more or less applications and examples of what I constrained myself to designate as the « General theory of the variable flow », feeling at that time already that the « Theory of Waterhammer », must be something very much different.

The scope of the present research is precisely this « Theory of Waterhammer », that is the investigation of the general laws of the phenomena grouped under this name, occurring in pipes which connect a reservoir of constant level to a gate mechanism of adjustable efflux, and of those laws which will permit a rational classification of the stated phenomena as well as of the tubes in which they occur.

Three fundamental principles characterize the method of this study and distinguish it from others previously published on this subject, i. e.:

1. It is contemplated to introduce not the absolute values, but the relative values of the unknown variable quantities referred to their initial values;

2. Not the pressure height but the corresponding velocity of efflux will be selected as the basic variable, or more accurately the relative value of the efflux velocity (that is the square root of the relative value of the pressure height);

3. As unit of time the duration of phase will be introduced, eliminating all considerations depending on the length of the conduit.

With the help of such simplifications, the laws of the phenomena of waterhammer appear as functions of only two variables, one of which (designated as the caracteristic of the pipeline and indicated by the symbol ρ) defines the pipe in normal flow conditions, and the other, indicated by symbol θ, defines the velocity of the gate operation. These laws will therefore be capable of simple graphical representation, and each of these graphs will embrace all possible pipelines and all possible speeds of gate operation.

The numerous (*) tests made since the publication of my first monograph absolutely verify the correspondence of the actual phenomena and the results of the variable flow, conferring especial importance to the conclusions of the study which follows, the generality of which conclusions, expressed by formulas or simple and synthetic graphs furnish accurate and practical criterious for their technical application.

In the following discussion it is assumed that the reader has a clear understanding of the significance of formulas (1), (2) and (3), previously given.

(*) I refer especially to the tests made by the engineers of Messrs. Picard Pictet e Co. on the corduits of the Ackersand, and to those reported by Prof. Neeser in the Bulletin technique de la Suisse Romande (Jan. 1910).

§ 2. — Fundamental formulas.

Considering a gate operation which, beginning at $t=0$ disturbs the regime of normal flow (defined by y_0 and v_0), and indicating by t_1, a time $< \mu$ (that is an instant of the 1st phase or the phase of the direct blow) and distinguishing, moreover, by indices 1, 2, 3, 4, etc, the values of the several variables corresponding to the instants

$$t_1, \quad t_1 + \mu, \quad t_1 + 2\mu, \quad t_1 + 3\mu, \text{ etc.},$$

which fall respectively in the 1st, 2nd, 3rd, 4th, phases, etc., we have, by formulas (2) and (2 *bis*), at the section near the gate (adopting capital letters for the variables refering to $x = 0$).

$$\left.\begin{aligned} Y_1 &= y_0 + F_1 \\ Y_2 &= y_0 + F_2 - F_1 \\ Y_3 &= y_0 + F_3 - F_2 \\ &\ldots\ldots\ldots \end{aligned}\right\} \qquad (4)$$

$$\left.\begin{aligned} V_1 &= v_0 - \frac{g}{a} F_1 \\ V_2 &= v_0 - \frac{g}{a}(F_1 + F_2) \\ V_3 &= v_0 - \frac{g}{a}(F_2 + F_3) \\ &\ldots\ldots\ldots \end{aligned}\right\} \qquad (5)$$

The first deduction of general character, resulting from the form of the sets of equations (4) and (5), is expressed by the fact that the series of the values of the pressures

$$Y_1, Y_2, Y_3, Y_4, \text{ etc.};$$

and that the series of values of the velocities

$$V_1, V_2, V_3, V_4, \text{ etc.},$$

separated from each other by time intervals equal to the duration of the phase, constitute interlocking series, that is series of values which depend only on initial conditions and on the positions of the gate at the instants

$$t_1, \quad t_1 + \mu, \quad t_1 + 2\mu, \quad t_1 + 3\mu, \text{ etc.},$$

but do not depend on the intermediate positions which the gate might have occupied, nor on the values of Y and V which the pressure or velocity might have taken in the intervals separating these instants.

The analitical expression of this serial interlocking will be obtained by eliminating the F's from the systems (4) and (5).

Adding each equation of (4) to the preceding one, and subtracting each equation of (5) from the preceding one we easily obtain.

$$\left.\begin{aligned} Y_1 - y_0 &= \frac{a}{g}(v_0 - V_1) \\ Y_1 + Y_2 - 2y_0 &= \frac{a}{g}(V_1 - V_2) \\ Y_2 + Y_3 - 2y_0 &= \frac{a}{g}(V_2 - V_3) \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad (6)$$

Denoting by ψ the ratio between the variable gate opening and the section of the conduit, we have

$$\psi_0 = \frac{v_0}{u_0} \qquad \text{et} \qquad \psi_i = \frac{V_i}{u_i}$$

where u_0 and u_i denote the efflux corresponding to the pressure heights y_0 and Y_i, and denoting by η the ratio $\frac{\psi}{\psi_0}$, that is the ratio of the (variable) rate of opening ψ to its original value ψ_0, in such a way that

$$\eta_0 = 1 \qquad \eta_i = \frac{\psi_i}{\psi_0} = \frac{V_i}{u_i} : \frac{v_0}{u_0}.$$

and putting

$$y_0 = \frac{u_0^2}{2g}\,; \quad Y_i = \frac{u_i^2}{2g}\,; \quad V_i = \eta_i\, u_i \frac{v_0}{u_0}$$

and introducting that characteristic ρ of the conduit, already mentioned in the preceding paragraph, defined by

$$\rho = \frac{av_0}{2g\,y_0} = \frac{av_0}{u_0^2}\,,$$

there is obtained

$$\left.\begin{aligned} u_1^2 - u_0^2 &= 2\rho\, u_0 (u_0 - \eta_1 u_1) \\ u_1^2 + u_2^2 - 2u_0^2 &= 2\rho\, u_0 (\eta_1 u_1 - \eta_2 u_2) \\ u_2^2 + u_3^2 - 2u_0^2 &= 2\rho\, u_0 (\eta_2 u_2 - \eta_3 u_3) \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad (8)$$

that is a system of quadratic equations, where the only unknowns are the efflux velocities u.

In order to be even more general, let us consider as the unknown, as already observed, not the absolute values of the efflux velocities, but their relative values referred to the initial value u_0 dividing therefore, equations (8) by u^2_0, and putting $\zeta_i = \frac{u_i}{u_0}$, we obtain

$$\left.\begin{aligned} \zeta^2_1 - 1 &= 2\rho\,(1 \quad - \eta_1\zeta_1) \\ \zeta^2_1 + \zeta^2_2 - 2 &= 2\rho\,(\eta_1\zeta_1 - \eta_2\zeta_2) \\ \zeta^2_2 + \zeta^2_3 - 2 &= 2\rho\,(\eta_2\zeta_2 - \eta_3\zeta_3) \\ \ldots\ldots&\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad (9)$$

the fundamental system for the study of the waterhammer phenomena, the theory of which, in fact, consists in the development of and deductions made from the equation system (9).

Observing that, by definition, $\eta_0 = 1$ and $\zeta_0 = 1$, the first equation of (9) can be written as

$$\zeta^2_0 + \zeta^2_1 - 2 = 2\rho\,(\eta_0\,\zeta_0 - \eta_1\zeta_1),$$

and that therefore the system (9) can be regarded as the result of the application of the general equation,

$$\zeta^2_{i-1} + \zeta^2_i - 2 = 2\rho\,(\eta_{i-1}\,\zeta_{i-1} - \eta_i\zeta_i)$$

which governs all the hydrodynamic phenomena which occur in a conduit fed by a reservoir of constant level and supplied with a gate at its other extremity.

The designation « characteristic of the conduit » given to the notation ρ is fully justified as in it are absorbed all the individual elements of the pipeline, such as the pressure height y_0, the normal velocity v_0, the diameter. thickness and elasticity of the pipe (included in the velocity of propagation a). The only element of the pipeline not accounted for in ρ is the length L, which enters into the definition of the duration of the phase $\mu = \frac{2\,L}{a}$, and which fixes the rythm of the *interlocked* series of the values ζ_i.

The relative intensity of the waterhammer phenomena, that is the unknown quantity which properly is the technical objective of a general theory, depends therefore, beside of the law of gate movement (value η), on the single characteristic ρ. It is, therefore, useful to investigate the meaning of this characteristic and the numerical limits (within the field of practical applications) of the values of same.

§ 3. — The characteristic ρ.

The intrinsic nature of the phenomena of the variable (or perturbed) motion of a liquid in a pipe, which motion is characterized by the continuous variation of both pressure and velocity, can be evidently defined as the result of the effect of such variations continuously transforming kinetic energy (of the moving liquid) into potential energy (elastic compression of the liquid and the elastic expansion of the pipe), and vice versa.

It is therefore legitimate to assume that the laws of such phenomena must bear a direct relation to the quantity of energy, both kinetic and potential, contained in each portion of the tube at any instant, and more particularly, that the characteristic $\rho = \frac{a\,v_0}{2\,g y_0}$ which defines the normal state of the pipeline must be in direct and close relation with the quantity of energy, kinetic and potential, which is contained in each unit of length of the conduit when the flow is permanent. And, in fact, it can be demonstrated that the value ρ is exactly equal to the half of the square root of the ratio of these two quantities of energy.

Let W_0 be the kinetic energy per unit length of the tube when the flow is normal, that is the kinetic energy possessed by a unit length of the water column flowing with a velocity v_0; and let W be the quantity of potential energy per unit length of the tube in normal state, that is the quantity of energy absorbed by the elastic compression of a unit length of water column and by the elastic expansion of a unit length of tube.

Using the accepted notations, it can be seen that

$$W_0 = \frac{\pi\,\omega\,D^2}{4} \cdot \frac{v_0^2}{2g} = \frac{\pi\,\omega\,D^2 v^2_0}{8g}.$$

Putting, moreover, $W = W' + W''$, were W' and W'' are the quantity of energy absorbed by the elastic compression of the liquid and by the elastic expansion of the tube, respectively, we have

$$W' = \frac{1}{2}\frac{\pi\,D^2}{4}\frac{\omega\,y_0}{\varepsilon} \cdot \omega\,y_0 = \frac{\pi\,\omega^2 D^2 y^2_0}{8\,\varepsilon},$$

$$W' = \frac{1}{2} \cdot \frac{\omega\,y_0 D}{2} \cdot \frac{\omega\,y_0\,D}{2e}\,\frac{\pi\,D}{E} = \frac{\pi\,\omega^2\,D^2 y^2_0}{8\,E} \cdot \frac{D}{e},$$

which, added, give

$$W = W' + W'' = \frac{\pi\,\omega^2\,D^2\,y^2_0}{8}\left(\frac{1}{\varepsilon} + \frac{1}{E} \cdot \frac{D}{e}\right)$$

and, by equation (1)

$$W = \frac{\pi\,\omega\,g\,D^2\,y^2_0}{8\,a^2}.$$

Dividing W_0 by W, we obtain

$$\frac{W_0}{W} = \left(\frac{a\,v_0}{g\,y_0}\right)^2 = 4\,\rho^2\,;$$

or

$$\rho = \frac{1}{2}\sqrt{\frac{W_0}{W}}, \qquad (10)$$

as stated above.

Having made clear the intrinsic significance of the characteristic ρ, we now will see between what limits its numerical value ranges within the field of practical applications.

I have observed in § 1 that, based on equation (1 *bis*), the value of a will range from a minimum of from 600 to 700 m. for thin riveted pipes of large diameters to a maximun of from 1200 to 1300 m. for thick pipes of small diameters.

As, on the other hand, the normal velocity v_0 ranges between 1,50 m. and 3 m., it is easy to conclude that, if the thickness of the pipes is determined by methods in general use, that the value of ρ will vary from:

a minimum of $\rho \backsim 0{,}10$ for high heads and small velocities (for example $y_0 \backsim 1000$ m. $v \backsim 1{,}50$ m.)

to a maximum of $\rho \backsim 10$ for low heads and large velocities, (for example $y_0 \backsim 10$ m.; $v_0 \backsim 3$ m.).

Before passing of the numerical calculation of ρ, it is necessary to make a preliminary observation.

All the formulas heretofore discussed and those which will follow assume that the velocity of propagation a, and therefore also the ratio $\frac{D}{e}$ are constant along the pipeline, this being the condition of integrability of the differential equations of the variable motion from which the fundamental equations (2) are derived.

This assumption is substantially correct in many cases (for example for the long pipelines of city water supplies) but is seldom true in the case of pipelines feeding power generating machinery.

It is at once evident that such conduits are mostly laid on a slope; consequently the ratio $\frac{D}{e}$ (e being the thickness determined on the basis of the static head obtaining at each section of the conduit) will be diminishing from upstream to downstream, an therefore the value a will diminish from the gate toward the reservoir.

It will be convenient, in such a case, to introduce an average value for a corresponding to the total time $\Sigma\left(\frac{l_x}{a_x}\right)$ which a pressure variation needs to travel the several portions, l_x, of the pipeline with a variable velocity a_x; and it is legitimate to assume such an average value of a for the calculation of the characteristic ρ of the tube, if phenomena of such duration are investigated which permit of the elasticity of the whole pipe entering into the game.

This will certainly be the case when we deal with phenomena the duratiou of which considerably exceeds the duration of the phase $\mu = \frac{2L}{a}$, that is, with the phenomena in the phases of the counterblow, while for phenomena of a duration, $< \mu$, that is, during the phase of the direct blow, we will have to assume for a, in the calculation of the characteristic ρ, that velocity of propagation which corresponds to the lower end of the tube. A typical illustration of this latter case is the sudden closure of the gate.

Reserving for subsequent studies a further discussion of this question, it can he concluded that the variation of a ranges only between such limits, that the substitution of an average value for *same* will have no sensible influence upon the accuracy of the results. In order to give a more concrete idea of such limits, we will discuss the case of a riveted steel pipeline of constant slope, assuming that the thickness of the shell, at each point, is calculated on the basis of the static head and on the basis of a constant coefficient of resistance $R = 10^6$ kg/mq and by adding a constant which we will assume to be proportional to the diameter, being 0,0025 m. per m. of D.

Indicating, moreover, by y_x the static head in any section of the pipeline of abcissa x, we have

$$\frac{2e}{D} = \frac{1000.\, y_x}{R.\, 10^6} + 0{,}005,$$

which, substitute in (1 *bis*), and making $R = 7$ kg./mm² gives for the velocity of propagation a_x, in the portion of the pipe of abcissa x, with great approximation

$$a_x = 1425 \sqrt{\frac{y_x + 35}{y_x + 180}} \qquad (11)$$

by the help of which te following table is calculated:

y_x	a_x	y_x	a_x	y_x	a_x
0	628	90	974	350	1216
10	703	100	990	400	1236
20	747	120	1025	500	1266
30	792	140	1054	600	1287
40	834	160	1080	700	1305
50	865	180	1102	800	1318
60	897	200	1122	900	1329
70	924	250	1161	1000	1338
80	948	300	1192	1100	1346

By the help of these values, and applying the relation

$$\frac{L}{a} = \Sigma \left(\frac{l_x}{a_x} \right)$$

we obtain the following table of values of the average velocity of propagation and of the values of the average characteristic ρ for $v_0 = 1{,}5$ m. and $v_0 = 3$ m.

for pipes of different heads, arranged in decressing order of the heads:

y_0	a	Values of ρ		y_0	a	Values of ρ	
		$v_0 = 1{,}5$	$v_0 = 3$			$v_0 = 1{,}5$	$v_0 = 3$
1000	1184	0,09	0.18	120	861	0,55	1,10
800	1153	0,10	0,21	100	837	0,64	1,28
600	1110	0,14	0,28	80	809	0,77	1,55
500	1082	0,16	0,33	60	777	0,99	1,98
400	1046	0,20	0,40	40	739	1,41	2,83
800	1000	0,25	0,61	30	717	1,83	3,66
200	936	0,36	0,72	20	694	2,65	5,30
140	883	0,48	0,97	10	665	5,09	10,18

which values, as can be seen, show that ρ ranges practically between 0.10 and 10, as previously stated.

As a rule, I will use, in numerical examples relative to hydraulic power plants, such values of ρ which are of the order of magnitude embraced by this table.

These systems of values, moreover, are susceptible of easy and simple graphic representation by means of a straight line diagram. As

$$\rho = \frac{a\, v_0}{2 g y_0}$$

the characteristic ρ can be represented, as a function of v_0, in a cartesian system of coordinates (Fig. 1), by a straight line passing through the origin; the angular coefficient of this line will depend on y_0 and a, or, in ultimate analysis, on the normal head y_0, so that the complete system of the values of ρ, for all possible conduits (that is for all y_0) and for all assumed normal velocities v_0, will be represented by a set of straight lines radiating from the origin of the axes ρ and v_0; some of these lines are drawn on Fig. 1, which does not need more explanation.

The great importance of the cartesian coordinates (see § 5) in the Theory of Waterhammer confers a specific utility to this diagram, to which, in the following study frequent reference will be made.

Refering to formula (10), let us finally observe that, as the characteristic ρ can assume values between 0.10 and 10, the ratio $W_0 : W$ of the kinetic and potential energy of the pipeline can correspondingly assume values from 0.04 to 400, that is they can vary between limits which may make their ratio 1 : 10000.

This statement will make it easy to understand why phenomena of the waterhammer, which are functions of the relative values of these two quantities of energy, are susceptible of attaining very different relative values and of following very different laws, according to whether the head is large or small. The direct proof of these facts will be given when the laws of water-hammer produced by diverse gate operation will be discussed.

§ 4. — Discontinuity of the law of variation ot the pressure

If one operates the efflux gate (opening or closing) in a continuous way, but in such a manner that at the beginning of the operation the variation of the rate of opening $\frac{\delta \eta}{\delta t}$ is not zero, a sudden discontinuity in the law of pressure variation occurs corresponding to the instants $t = \mu,\ 2\mu,\ 3\mu$, etc., that is at the end of the phase of the direct blow and of all successive phases of the counterblow.

Differentiating equations (9) with respect to the time, we obtain

$$\left.\begin{aligned} (\zeta_1 + \rho\eta_1)\frac{\delta\zeta_1}{\delta t} &= -\rho\zeta_1\frac{\delta\eta_1}{\delta t} \\ (\zeta_1 - \rho\eta_1)\frac{\delta\zeta_1}{\delta t} + (\zeta_2 + \rho\eta_2)\frac{\delta\zeta_2}{\delta t} &= \rho\left(\zeta_1\frac{\delta\eta_1}{\delta t} - \zeta_2\frac{\delta\eta_2}{\delta t}\right) \\ (\zeta_2 - \rho\eta_2)\frac{\delta\zeta_2}{\delta t} + (\zeta_3 + \rho\eta_3)\frac{\delta\zeta_3}{\delta t} &= \rho\left(\zeta_2\frac{\delta\eta_2}{\delta t} - \zeta_3\frac{\delta\eta_3}{\delta t}\right) \end{aligned}\right\} \quad (12)$$

in which the symbols with the index 1 represent values relating to an instant $t = t_1$, where $0 < t_1 < \mu$, and symbols with indices 2, 3, 4, etc., represent values relating to the instants

$$t = t_1 + \mu \qquad t = t_1 + 2\mu \qquad t = t_1 + 3\mu,\ \text{etc.}$$

that is the system (12) expresses the relation between the values η, ζ, and their derivatives at instants separated by the time interval of the phase.

Applying this system (12) to the series of values

$$t = 0, \qquad t = \mu, \qquad t = 2\mu,\ \text{etc.},$$

which we will designate as instants of the « total rythme », it is evident that at the instant $t = i\mu$, which separates the i^{th} phase from the $i + 1^{th}$, two distinct values of $\frac{\delta\zeta}{\delta t}$ can be considered, i. e.:

the value $\frac{\delta\zeta_i}{\delta t}$ corresponding to the last instant of the i^{th} phase

and » » $\frac{\delta\zeta_{i+1}}{\delta t}$ » » first » » » $i + 1^{th}$ »

We will demonstrate that these two values are always different, even assuming a continuous law of operation, if the value of $\frac{\delta\eta}{\delta t}$ is different from zero at the start of the operation.

In this study let us assume that the symbols

$$\eta_1,\ \eta_2,\ \eta_3,\ \text{etc},$$
$$\zeta_1,\ \zeta_2,\ \zeta_3,\ \text{etc.},$$

represent the values of η and ζ with respect to the instants of the total rythme, and we will designate moreover, by

$$\left(\frac{\delta\eta}{\delta t}\right)_0\quad \left(\frac{\delta\eta}{\delta t}\right)_1\quad \left(\frac{\delta\eta}{\delta t}\right)_2,\ \text{etc.};$$

the values of the variation of the η at the instants of the total rythme;

$$\text{by } \left(\frac{\delta\zeta_1}{\delta t}\right)_0 \text{ and } \left(\frac{\delta\zeta_1}{\delta t}\right)_1,$$

the values of the variation of ζ at the beginning and at the end of the first phase;

$$\text{by } \left(\frac{\delta\zeta_2}{\delta t}\right)_1 \text{ and } \left(\frac{\delta\zeta_2}{\delta t}\right)_2,$$

the values of the variation of ζ at the beginning and at the end of the second phase; etc.

In order to apply the systems (12) to the interlocking series of te values corresponding to the first instants of each phase; that is, in the instants

$$t = 0 \quad t = \mu \quad t = 2\mu, \text{ etc },$$

considered as the first instants of the

1st 2nd 3rd phases, etc.;

we must substitute evidently for

$$t = 0$$

the symbols of the total rythme

$$\eta_0 \qquad \zeta, \qquad \left(\frac{\delta\eta}{\delta t}\right)_0 \qquad \left(\frac{\delta\zeta_1}{\delta t}\right)_0,$$

in lieu of the symbols of the intermediate rythme

$$\eta_1 \qquad \zeta_1 \qquad \frac{\delta\eta_1}{\delta t} \qquad \frac{\delta\zeta_2}{\delta t},$$

for

$$t = \mu,$$

the symbols

$$\eta_1 \qquad \zeta_1 \qquad \left(\frac{\delta\eta}{\delta t}\right)_1 \qquad \left(\frac{\delta\zeta_2}{\delta t}\right)_1,$$

for the symbols

$$\eta_2 \qquad \zeta_2 \qquad \frac{\delta\eta_2}{\delta t} \qquad \frac{\delta\zeta_2}{\delta t},$$

for

$$t = 2\mu,$$

the symbols

$$\eta_2 \qquad \zeta_2 \qquad \left(\frac{\delta\eta}{\delta t}\right)_2 \qquad \left(\frac{\delta\zeta_3}{\delta t}\right)_2,$$

for the symbols

$$\eta_3 \qquad \zeta_3 \qquad \frac{\delta\eta_3}{\delta t} \qquad \frac{\delta\zeta_3}{\delta t},$$

and so on.

Observing that $\eta_0 = \zeta_0 = 1$, the system (12) becomes

$$(1 + \rho)\left(\frac{\delta\zeta_1}{\delta t}\right)_0 = \qquad -\rho\left(\frac{\delta\eta}{\delta t}\right)_0,$$

$$(1 - \rho)\left(\frac{\delta\zeta_1}{\delta t}\right)_0 + (\zeta_1 + \rho\eta_1)\left(\frac{\delta\zeta_2}{\delta t}\right)_1 = \rho\left[\left(\frac{\delta\eta}{\delta t}\right)_0 - \zeta_1\left(\frac{\delta\eta}{\delta t}\right)_1\right]$$

$$(\zeta_1 - \rho\eta_1)\left(\frac{\delta\zeta_2}{\delta t}\right)_1 + (\zeta_2 + \rho\eta_2)\left(\frac{\delta\zeta_3}{\delta t}\right)_2 = \rho\left[\zeta_1\left(\frac{\delta\eta}{\delta t}\right)_1 - \zeta_2\left(\frac{\delta\eta}{\delta t}\right)_2\right], \text{ etc. (13)}$$

On the other hand, if we wish to apply the system (12) to the interlocked series of values corresponding to the last instants of each phase, that is to the instants,

$$t = \mu, \qquad t = 2\mu, \qquad t = 3\mu, \qquad \text{etc.},$$

conceived as the last instants of the

1st, 2nd, 3rd, etc.

phases, it is not necessary to change the indices of the η and ζ, except as to indicate that they are values of the total rythme, and we will substitute:
for

$$t = \mu$$

the symbols of the total rythme

$$\eta_1 \qquad \zeta_1 \qquad \left(\frac{\delta\eta}{\delta t}\right)_1 \qquad \left(\frac{\delta\zeta_1}{\delta t}\right)_1,$$

in lieu of the symbols of the intermediate rythme

$$\eta_1 \qquad \zeta_1 \qquad \frac{\delta\eta_1}{\delta t} \qquad \frac{\delta\zeta_1}{\delta t},$$

for

$$t = 2\mu,$$

the symbols

$$\eta_2 \qquad \zeta_2 \qquad \left(\frac{\delta\eta}{\delta t}\right)_2 \qquad \left(\frac{\delta\zeta_2}{\delta t}\right)_2,$$

for the symbols

$$\eta_2 \qquad \zeta_2 \qquad \frac{\delta\eta_2}{\delta t} \qquad \frac{\delta\zeta_2}{\delta t},$$

and so on.

The system (12) becomes therefore

$$(\zeta_1 + \rho\,\eta_1)\left(\frac{\delta\zeta_1}{\delta t}\right)_1 = \qquad - \rho\zeta_1\left(\frac{\delta\eta}{\delta t}\right)_1$$

$$(\zeta_1 - \rho\,\eta_1)\left(\frac{\delta\zeta_1}{\delta t}\right)_1 + (\zeta_2 + \rho\,\eta_2)\left(\frac{\delta\zeta_2}{\delta t}\right)_2 = \rho\left[\zeta_1\left(\frac{\delta\eta}{\delta t}\right)_1 - \zeta_2\left(\frac{\delta\eta}{\delta t}\right)_2\right], \text{ etc.} \quad (14)$$

Comparing (13) and (14) it is evident that

$$\left(\frac{\delta\zeta_1}{\delta t}\right)_1 = \text{the value of } \frac{\delta\zeta}{\delta t}$$

at the end of the 1st phase from the 1st of (14) and

$$\left(\frac{\delta\zeta_2}{\delta t}\right)_2 = \text{the value of } \frac{\delta\zeta}{\delta t}$$

at the beginning the 2nd phase from the 2nd of (13), are numerically different values with the single exception of the case that at the beginning of the operation $\left(\frac{\delta\eta}{\delta t}\right)_0 = 0$, in which case the 2nd of (13) becomes

$$(\zeta_1 + \rho\,\eta_1)\left(\frac{\delta\zeta_2}{\delta t}\right)_1 = - \rho\,\zeta_1\left(\frac{\delta\eta}{\delta t}\right)_1,$$

which compared with the Ist of (14) shows that

$$\left(\frac{\delta \zeta_1}{\delta t}\right)_1 = \left(\frac{\delta \zeta_2}{\delta t}\right)_1,$$

and in general

$$\left(\frac{\delta \zeta_i}{\delta t}\right)_i = \left(\frac{\delta \zeta_{i+1}}{\delta t}\right)_i.$$

In the following study, I shall exclude this latter assumption, observing that the variation of η is sensibly constant as a rule in pipelines feeding hydro-electric plants to which special reference is here made.

We will therefore assume that the variation of η is linear and in all what follows we will consider exclusively such series of values the law of variations of which presents discontinuities at the instants of the total rythme.

It evidently follows that if a diagram of the pressure heights $Y_i = \zeta^2_i\, y_0$ is constructed as a function of t for abcissa, this will present a broken line the vertices of which will correspond to the instants of the total rythme

$$t = 0, \quad \mu, \quad 2\mu, \quad 3\mu, \quad \text{etc.},$$

having respectively the ordinates

$$y_0, \quad Y_1, \quad Y_2, \quad Y_3, \quad \text{etc.}$$

Excepting certain special categories of pipelines, the investigation of which is reserved for later study, the maximum and minimum values of the pressure will therefore occur at such instants of the total rythme, and we can say that we have, in general, adequate knowledge of the laws of waterhammer for a given method of operation, if we know the series of pressure heights of the total rythme Y_1, Y_2, Y_3, etc., which are generated near the gate section.

We will be satisfied, for the moment, with the discussed general statements of the periodical discontinuity of the laws of watherhammer to which we will return at the special studies of the different laws of gate operation.

§ 5. — Synopsis of the phenomena of waterhammer represented in Cartesian Coordinates.

It results from the fundamental equation (9), as already remarked, that the laws and the percentual intensity of the waterhammer phenomena depend exclusively on the characteristic ρ (in which are absorbed, with te exception of the length L, all individual elements of the pipeline) and on the operation of the gate, that is on the series of the proportional gate opening η_1, η_2, η_3, etc., which occur at time intervals μ.

Two pipelines having the same characteristic ρ, but different lengths and velocities of propagation, will therefore show identical phenomena of waterhammer if they are operated in such a manner as to produce equal values η_1, η_2, η_3, etc., of the rate of gate opening at homologuous phase intervals.

I propose to consider such pipelines and also the movements of their gates as identical, notwithstanding that such operations had to be executed evidently at different speeds proportional to the respective duration of the phases. It can be seen at once that the assumption of the duration of the phase as the unit of time simplifies and synthetizes the whole problem, in other words it is equivalent to assigning to all pipelines the same common length.

This is the third of the fundamental principles of the theory, stated in § 1, constituting the key of graphical representation of the phenomena of waterhammer, which I will designate by the name of cartesian synopsis.

We have already observed that the operation of the efflux openings, in conduits to which these theories will be applied, is executed with a sensibly constant speed, that is, the rate of variation of the efflux opening is sensibly uniform per unit of time during the operation (linear variation), and such speed is generally defined by indicating the total time τ necessary to produce complete closure reckoned from the state of regimen.

Assuming, therefore, the duration of the phase as the unit of time, and denoting by θ the total time of closure expressed in units of μ

$$\theta = \frac{\tau}{\mu} = \frac{a\,\tau}{2\,L}; \tag{15}$$

the series of the rates of opening η is defined by

$$\eta = 1 \mp \frac{t}{\theta}, \tag{16}$$

in which t is measured by the same time unit μ.

If the nature of the operation (closure, opening or alternating motion), and the time θ are given, all functions of the pipeline during the perturbed regime are determined, and all elements for the application of the fundamental system (9) which govern the laws of the phenomena are known.

We can therefore state:

If the operation of the gate is executed at a uniform speed, the laws of the waterhammer produced are functions of the two parameters ρ (characteristic) and θ (a numeral defining the speed of the operation).

The reader should familiarize himself with the conception that a pipeline, insofar the phenomena of waterhammer are concerned, is completely characterized by the two parameters ρ and θ and that such pair of parameters, in reality, represent a triple infinity of identical conduits from the point of view of waterhammer, that is, the triple infinity of conduits of which the five individual elements, i. e.,

y_0 = pressure height
a = velocity of propagation of the variable pressures
L = length of conduit
v_0 = normal velocity of flow
τ = time defining the speed of gate operation

satisfy the two conditions:

$$\frac{av_0}{2g\,y_0} = \rho \qquad \frac{a\tau}{2L} = \theta\,.$$

If the thickness of the pipe, as is ordinarily the case, is determined on the basis of the static head y_0, the velocity a becomes a function of y_0 (See

§ 3) and the total of pipelines represented by the two parameters ρ and θ reduce to a double infinity.

The reader should, by means of numerical examples, familiarize himself with the idea that this triple (or double) infinity of pipelines represented by the said pair of parameters comprise very different conduits, although always being within the limits of practical applications.

For example, the two conduits given by the following elements:

1° $y_0 = 300$ m $a = 1000$ m/sec L = 900 m $v_0 = 3{,}60$ m/sec $\tau = 9$ sec

2° $y_0 = 100$ m $a = 840$ m/sec L = 300 m $v_0 = 1{,}43$ m/sec $\tau = 3{,}6$ sec

are both represented by the parameters:

$$\rho = 0{,}60 \qquad \theta = 5.$$

It should be observed moreover, that of the five elements which characterize a conduit under pressure, the first three, i. e.: y_0, a, L are constructive and in general invariable, whereas v_0 and τ are functional and are frequently modified, varying the flow of the pipe and the speed of the operation. By such modification of either of these two latter elements the pipe becomes a new conduit, in which the phenomena of waterhammer may present entirely different laws from those applying when the original assumptions are considered.

Assigning therefore to the parameters ρ and θ all possible values between the limits of practical applications, a double infinity of conduits is obtained in which all possible pipelines are comprised and all possible speeds of gate operation; but in reality, each of the conduits so defined by the pair of parameters (ρ, θ) represents a triple (or double) infinity of conduits which will follow identical laws and in which the waterhammer will reach the same relative values.

It is this grouping of all the imaginable conduits which makes possible the systematic and complete study of waterhammer and the full utilization of the theory.

The Cartesian Synopsis, the conception of which directly follows from the preceding considerations, is a valuable instrument for the investigation and representation of waterhammer phenomena. Assuming two cartesian coordinate axes ρ and θ (See Fig. 2), it is obvious that all the laws of the waterhammer phenomena, that is all the deductions and conclusions both of finite and differential character, which can be drawn from the fundamental system (9), can be represented graphically in the positive quadrant of the system (ρ, θ) which graph, in the following, I have designated as the *Cartesian Synopsis of pipelines*.

A point with the coordinates (ρ, θ) represents therefore a certain pipeline, or rather the triple infinity of pipelines defined by the parameters ρ and θ, and the positive quadrant evidently contains all possible conduits.

We may distinguish:

conduits situated at a point of the synopsis; this point indicating a pipeline the constructive and functional elements of which correspond to the parameters ρ and θ; and conduits situated on a line or in a zone of the synopsis, which will indicate conduits, the elements and consequently the parameters of which vary between certain given limits.

For example, if one considers a pipeline, the initial flow of which can vary between certain limits (with a corresponding variation of v_0 and consequently of ρ) this pipeline will be represented in the synopsis not by a point anymore, but by a line A B (Fig. 2) parallel with axis ρ. Conversely, a pipeline with a constant initial discharge, the gate of which is operated with different speeds, will be represented by a line A C parallel with axis θ.

Finally, a conduit which can function with variable flow and variable speed of gate operation will be represented in the synopsis by the area of the rectangle corresponding to the limiting values of ρ and θ.

We will adopt, as a rule. (See Fig. 2) the vertical axis as the θ axis, values being measured downward. and the horizontal axis as te ρ axis, placing above same the diagram of the characteristic (Fig. 1), which immediately furnishes all the values of y_0 and v_0, corresponding (for the case of a riveted conduit) to any given value of ρ.

The selection of the scales of the values ρ and θ is evidently arbitrary, but it will be found convenient to select a larger scale for ρ and a smaller one for θ, as, if it is contemplated to represent all practically possible pipelines, it is necessary to consider values of ρ equal to or little greater than 10 (See Table in § 3), and values of θ ranging to about 25 or 30. We will see that, while te laws of waterhammer change sensibly from one zone of conduits to the next in the sense of ρ, the contrary is true for changes in the sense of θ.

I wish to remark also, that while the quadrant of the synopsis embraces all the possible conduits, the several zones have different importance in a technical sense, and some of them have no importance *at all*.

We can almost *absolutely* exclude the assumption that pipelines exist wich are characterized by small values of ρ and large values of θ, or large values of ρ and small values of θ.

The conduits characterized by small values of ρ are evidently those of high and very high heads (See Fig. 1) which naturally necessitate long pipelines and therefore a large value of the duration of phase, so that even if the gate operation is slow, it will result in small values of θ. On the contrary, the pipelines characterized by large values of ρ are those of low heads, which, excepting special cases, need only short lengths of pipelines, that will have short durations of the phase, so that even if the gate operation is rapid, they will result in large values of θ.

Use of the cartesian synopsis.

The cartesian synopsis can be utilized either as a diagram to find numerical values or as a classifying diagram, but such uses can be only generally shown at present, as a complete exposition cannot be made without invading the field of special researches which will form the subject of the subsequent notes.

We can, by help of the fundamental system (9), indicate in the plan of the synopsis those curves which are the loci of pipelines for which the pratically interesting phenomena to be determined range within certain given ratios, and we can thereby plot valuable diagrams, for the quick investigation of any conduit with respect to these phenomena

For example, the hyperbolic branch shown on the synopsis (Fig. 2) is the locus of the conduits for which the ratio of the pressure of the direct blow, during a closing operation, equals the value

$$\zeta^2_1 = \frac{Y_1}{y_0} = 1{,}5\,;$$

and it is obvious that by drawing a series of similar curves for a series of the values $\zeta^2{}_1$, we can construct a diagram of the pressures of the direct blow in closure.

But we also can, always by the system (9), demonstrate that certain zones of the synopsis (that is the conduits which they represent), show, for given gate operations, certain particular characteristics, interesting from the point of view of technical application; it is easy to isolate the zones in which these phenomena are produced, so that our cartesian synopsis can effectively serve as a classificative diagram of the waterhammer phenomena.

For example, the hyperbolic branch *s* (Fig. 2) divides the plan into two zones; that on the left contains all the conduits for which the maximum pressure of closure is due to the direct blow, wile the zone situated to the right of *s* contains all conduits for which the maximum waterhammer of closure is one of the pressures of the counterblow.

§ 6. — The circular diagram of the interlocked series.

Another useful instrument for the investigation of the waterhammer phenomena is furnished by a rather simple metod of calculating graphically, by means of circular diagrams, the interlocked series of values $\zeta_1, \zeta_2, \zeta_3, \zeta_4$, etc. of a given pipeline (that is a given value of ρ) and of a given gate operation that is a given series of the values $\eta_1, \eta_2, \eta_3, \eta_4$, etc.

This method of graphical calculation is derived from the fundamental equations (9), by observing that they can be interpreted as equations of circles.

Remembering that in our notation

$$\eta_0 = 1 \qquad \zeta_0 = 1\,,$$

the system of equations (9) can be written in the form

$$\left.\begin{aligned}
(\zeta_0 - \rho\eta_0)^2 + (\zeta_1 + \rho\eta_1)^2 &= \overline{\rho\eta_0}^2 + \overline{\rho\eta_1}^2 + 2\\
(\zeta_1 - \rho\eta_1)^2 + (\zeta_2 + \rho\eta_2)^2 &= \overline{\rho\eta_1}^2 + \overline{\rho\eta_2}^2 + 2\\
(\zeta_2 - \rho\eta_2)^2 + (\zeta_3 + \rho\eta_3)^2 &= \overline{\rho\eta_2}^2 + \overline{\rho\eta_3}^2 + 2\\
\text{etc.,} \qquad\qquad & \qquad \text{etc.}
\end{aligned}\right\} \qquad (17)$$

It is evident that interpreting the ζ's of even indices and the ζ's of uneven indices respectively as coordinates of a point referred to rectangular axes, equations (17) represent a series of circles, the coordinates of the centers and the radii of which are given by:

	Coord. of Centers	radii	
1°	$(+\rho\eta_0\,, -\rho\eta_1)$	$\sqrt{\overline{\rho\eta_0}^2 + \overline{\rho\eta_1}^2 + 2}$	
2°	$(+\rho\eta_1\,, -\rho\eta_2)$	$\sqrt{\overline{\rho\eta_1}^2 + \overline{\rho\eta_2}^2 + 2}$	(17 *bis*)
3°	$(+\rho\eta_2\,, -\rho\eta_3)$	$\sqrt{\overline{\rho\eta_2}^2 + \overline{\rho\eta_3}^2 + 2}$	
	etc.	etc.	

One can therefore draw such a series of circles, remembering that $\zeta_0 = 1$, and the successive values of

$$\zeta_1, \zeta_2, \zeta_3, \zeta_4, \text{ etc.},$$

of an interlocked series can be constructed for any law of the speed of gate operation, that is for the series of the rate of gate opening

$$\eta_1, \eta_2, \eta_3, \eta_4, \text{ etc.},$$

corresponding to the instants

$$t_1, \quad t_1 + \mu, \quad t_1 + 2\mu, \quad t_1 + 3\mu, \text{ etc.}$$

Taking, for this purpose, (See Fig. 3) the horizontal axis for the ζ 's of even indices, and the vertical axis for the ζ 's of uneven indices, then C_1, (absissa $= \rho\eta_0 = \rho$; ordinate $= -\rho\eta_1$) by the first of equations (17) is the center of the first circle. Draw line $C_1 O$ and measure upon a normal to same at O the value $OK_1 = \sqrt{2}$, then, by the first of (17 bis), $C_1 K_1$ evidently is the radius of the first circle, γ_1.

A normal to the horizontal axis at A_1, ($OA_1 = \zeta_0 = 1$) will cut the circle γ_1, in D_1, and the segment $A_1 D_1$, will be evidently $= \zeta_1$.

The second circle γ_2 will have its center at C_2 (coordinates $+\rho\eta_1, -\rho\eta_2$) and its radius will be $C_2 K_2$, determined by $OK_2 = \sqrt{2}$, this being drawn at right angles to $C_2 O$. This circle will cut the horizontal drawn from D_1 at D_2. By the 2nd equation of (17 bis) $\zeta_2 = A_2 D_2$.

Analogously the successive circles γ_3, γ_4, etc. can be drawn and the values ζ_3, ζ_4, etc. determined.

Such graphical proceeding therefore furnishes a quick solution of the system of equations (9) with respect to the unknown ζ, and it is pratical to use for this purpose a large scale, because, in practice, not the values ζ_i but those of ζ_i^2 are of interest, which values give the ratio of the pressure heights,

$$\zeta^2_i = \frac{u^2_i}{u^2_0} = \frac{Y_i}{y_0},$$

and as small errors in the values of ζ_i may become sensible in the values of ζ_i^2.

It should be observed, however, that the importance of such graphical proceeding does not lie in the quick solution of the system (9) for the determination of the values ζ^2 which define the variation of the pressure height for a given gate operation. As already stated, such is not the real object of the present research, which has in view the investigation of the general laws of the waterhammer phenomena, and the furnishing of rational criterions for the design of pipelines.

But, also from this point of view, the circular diagram of the interlocked series furnishes an accurate instrument of research, because its geometrical properties, as depending upon the method of gate operation (the series of relative gate openings η_1, η_2, η_3) furnish an elementary method for the demonstration of interesting properties of the laws of the waterhammer phenomena.

§ 7. — Technical problems and program of subsequent researches.

The preceding discussion has clearly shown how the study of the phenomena of the variable motion in pipelines — the theory of waterhammer — can be systematically conducted by the help of the system of the fundamental equations (9) and their derivatives, also by the circular diagram of the interlocked series derived from the system (9).

We shall now define the field and method of the succeeding researches.

We shall limit ourselves to typical problems and assumption of the linear law of operating the gate, and first of all, will treat in detail the research of the laws of the variable pressures in four principal cases:

(*a*) For the operation of closure, assuming that the proportionate gate openings η_1, η_2, η_3 constitute a series of values linearly decreasing from unity

(*b*) For the operation of opening, assuming that the proportionate gate openings η_1, η_2, η_3 constitute a series of values linearly increasing from unity (eventually from zero);

(*c*) For an alternate closing and opening, with the same rythme as the duration of the phase, in such a way that the proportionate gate openings will regain the same value at each alternate interval of the phase, which results in the phenomenon called the resonance;

(*d*) For the stationary gate, assuming that η_1, η_2, η_3 constant, after the flow was disturbed by a preceding operation.

We will treat, first of all, each of these problems by the graphic method, by means of the circular diagram of the interlocked series; this method, wich, in a sense, constitutes, a graphic theory of the waterhammer, will permit, as already observed, to point out some of the more outstanding features of these phenomena.

We then will start upon the analytical discussion of these problems, and will determine, by help of the cartesian synopsis (See § 5) the general laws which classify the double (or quintuple) infinity of the possible pipelines with respect to each of these four kinds of operation.

The reader will be able to see that these problems which are seemingly extremely complicated, are susceptible to elegantly simple solutions; for example, the drawing of a single circle of the diagram of the interlocked serie gives the six limiting values of technical interest of the variable pressure, i. e., two each for the three movements of closure, opening and alternate operation of the gate.

Moreover, the diagrams which we will derive from the cartesian synopsis will permit to quickly solve the problems, concerning the rational design of conduits (the selection of the arbitrary elements) and also the finding of those gate operations which produce dangerous effects and the selection of the limiting conditions for each category of pipelines.

These studies will be folloved by a study of the law of propagation of the instantaneous pressures along the conduit and of the influence of the variable diameter in reducing or iucreasing the intensity of the propagated pressure.

Finally, the problems relative to the law of the variation of the kinetic-energy of the fluid jet in closing or opening will form a second class of exhaustive analytical studies, the knowledge of which laws will determine rational criterions for the regulation of the turbines.

In my preceding monograph I have already pointed out the conditions under which the kinetic energy of the fluid jet can increase in the first instants of a closing operation. I will extend, in a more general way, this study to the whole phase of the direct blow and the successive phases of the counterblow and will point out, with the help of the cartesian synopsis, the law of these little known, or entirely unknown, phenomena.

It will result from this research that for large enough values of the characteristic (that is for small heads) the increase of the kinetic energy of the fluid jet in closure may make it difficult, or even impossible to regulate the turbines in the ordinary way, which inconvenience can only be partly remedied by the addition of synchronously acting relief valves.

We will finally show how similar phenomena can be partially decreased or suppressed by the use of piezometric tubes or airchambers, which latter were already discussed to some extent in my preceding paper.

NOTE II.

WATERHAMMER IN CLOSURE.

§ 8. — The circular diagrams of the interlocked series at closure.

The graphic method derived from the circular diagram of the interlocked series (see § 6) furnishes a very elegant solution of the problems of waterhammer arising when the gate is being closed and permits the rapid determination of the successive members $\zeta_1, \zeta_2, \zeta_3 \ldots \zeta_i$ of any interlocked series for a given conduit and method of operation; it also acquires a particular importance in view of the observation in § 4 which states that for an operation executed according to a linear law, the law of the pressure variation is subject to periodical discontinuities, and the pressures platted as functions of orthogonal co-ordinates, present themselves as a broken line, having vertices corresponding to the instants of the total rythme.

As already observed in § 4, the series of pressures of the total rythme, therefore, contain (except in special cases) maximum and minimum values of the pressure during the closing operation (at any rate values of close proximity to maximum and minimum), so that the rapid determination of the ζ_i of the total rythme by means of the circular diagram of the interlocked series constitutes undoubtedly an important, if not complete, illustration of the waterhammer due to such gate operation. In the following discussion I will therefore use the circular diagram of the interlocked series principally to determine the ζ_i of the total rythme, assuming that the closure is executed by the linear law.

Applying the graphic construction shown in Fig. 3 (§ 6) to such determination of the ζ_i, it is easily seen that as

$$\rho, \quad \rho\eta_1, \quad \rho\eta_2, \quad \rho\eta_3, \quad \text{etc.}$$

constitute a linearly decreasing series, the resulting diagram will have the folloving properties (fig. 4 and 7):

1st. the centers C_1, C_2, C_3, etc., of the successive circles, γ_1, γ_2, γ_3, etc. all lie on the same straight line, forming 45° with the axes and passing above the origin O.

2nd. The circles γ_1, γ_2, γ_3, etc. all pass throught the same point M of the bisectrix of the right angle formed by the axes (normal to the locus line of the C's), the equal co-ordinates of which point we will designate by ζ_m.

The first postulate is evident from the linearity of the series of values, which, by couples and opposite signs, give the co-ordinates of C_1, C_2, C_3 etc., successively, while the second can be easily demonstrated by proving that the distance $\overline{OM}$ is the same no matter which is the center C_i from which the circle γ_2 is drawn.

Consider, for instance, C_1 to be the center of circle γ_1, and remember the sequence of the construction (see § 6) which is as follows; on a normal to C_1 O, in O, measure a distance $\sqrt{2} = OK$, and drawn the circle γ_1, from the center C_1 and radius C_1 K, the intersection of which with the bisectrix will give the point M (fig. 4 and 7).

Let L be the intersection of the bisectrix with the line connecting the centers, then

$$\overline{LM}^2 = \overline{C_1M}^2 - (\overline{C_1O}^2 - \overline{LO})^2$$

and, because

$$\overline{C_1M}^2 = \overline{C_1K}^2 = \overline{C_1O}^2 + \overline{OK}^2 = \overline{C_1O}^2 + 2$$

the first equation will become

$$\overline{LM}^2 = \overline{LO}^2 + 2$$

which is independent of the position of C_1 on the connecting lines.

M being determined in this manner by means of the circle γ_1, it will be easy to draw the other circles with the centers C_2, C_3, C_4, etc. and the radii C_i M.

For the construction of the successive values of ζ_i, we follow the procedure of § 6, in drawing (see fig. 4 and 7) a vertical at the point of which the absissa is $\zeta_0 = 1$, the ordinate of the intersection of this line with the circle γ_1 being ζ_1; draw a horizontal through this point to intersect with γ_2, the absissa of the intersecting point being ζ_2; from this point again draw a vertical to intersect with γ_3 at a point the ordinate of which is equal ζ_3, and so on as illustrated in figs. 4 and 7.

It also results from these figures that the centers C_1, C_2, C_3, etc., being situated alternately on both sides of the foot L of the bisectrix, approach progressively this point L and that the lengths of the ζ_1, ζ_2, ζ_3, etc., tend toward a limiting length equal to ζ_m the ordinate of the point M.

In fact, the extreme points of the co-ordinates ζ_i (see the large scale detail figs. 4 bis and 7 bis) determine a rectangular polygon, the sides of which constantly diminish in length, and the summits of which progressively approach the point M. This geometrical configuration permits therefore to conclude:

That the interlocked series ζ_i tend to a limiting value ζ_m, and therefore, the interlocked series of the pressure heights $Y_i = \zeta_i^2 y_0$ tend toward a limiting value $Y_m = \zeta_m^2 y_0$.

In the following discussion I will refer exclusively to the relative (percentual) value of the pressure, but will designate by the word « pressures » the relative pressures ζ_i^2 or ζ_m^2.

The diagrams of the interlocked series permit, moreover, to deduct some important conclusions upon the law by which the pressure approaches the limiting value.

1st Case

Examining first the diagram fig. 4, drawn for $\rho < 1$ ($\rho = 0.5$ and $\theta = 4$), it is noted that the first center C_1 is located necessarily to the left of the vertical through M, from which it results that ζ_1 is always $> \zeta_m$.

As, however, the second center C_2 is located necessarily below the horizontal through M (we have evidently $\rho\,\eta_1 < \rho < 1 < \zeta_m$) it results that, on the contrary, $\zeta_2 < \zeta_m$.

It is easy to conclude that by analogy that the successive ζ_i will be alternatily $\gtrless \zeta_m$.

In the stated case therefore the law of pressure in closure is oscillatorily asymptotic to the limiting value, as shown on the diagram fig. 4, and it is clearly shown by fig. 4 that the stated law is always verified when $\rho < 1$ because on this assumption the centers C_i of uneven indices are located to the left of the vertical through M, and the center C_i of even indices are below the horizontal through M.

In this case, the first pressure of total rythme ζ_1^2 (the pressure of the direct blow) is the maximum of the series of pressures of total rythme due to closure, and may be considerably larger than the limiting pressure of ζ_m^2; this is easily demonstrated by drawing the diagrams of the interlocked series for small values of ρ and θ. Fig. 5 gives the values of ζ_1, and ζ_m for $\rho = 0.64$ and corresponding to 4 kinds of gate operations, defined respectively by $\theta = 4; 2; 1{,}33$ and 1 (*). Graphically, and with sufficient accuracy we get

for $\theta = 4$	$\zeta_1 = 1{,}10$	$\zeta_m = 1{,}08$	$\zeta_1^2 : \zeta_m^2 = 1{,}04$
» $= 2$	» $= 1{,}22$	» $= 1{,}17$	» $= 1{,}09$
» $= 1{,}33$	» $= 1{,}36$	» $= 1{,}27$	» $= 1{,}15$
» $= 1$	» $= 1{,}51$	» $= 1{,}37$	» $= 1{,}21$

The first pressure of the total rythme, which, as pointed out, is the largest of the series of the total rythme, can nevertheless, for small values of θ, be smaller than the pressure of the intermediate rythme which occur at the beginning of the second phase, but as the graphic method is not very appropriate for the investigation of this singularity, the reader is referred to the more complete discussion ot this problem in § 13.

2nd Case.

If, to the contrary, we examine a circular diagram, drawn for ρ substantially grealer than unity, as for instance that of fig. 7 (where $\rho = 2$, $\theta = 5$) it is observed that the first center C_1 is situated to the right of the vertical passing through M (whatever be the speed of the gate operation) in such a way that

$$\zeta_1 < \zeta_m .$$

Moreover, as the second center C_2 is situated above the horizzontal passing through M, there will result that $\zeta_2 < \zeta_m$.

But the third center C_3 being a little to the left of the vertical passing through M, we again will have $\zeta_3 > \zeta_m$, wile the successive ζ_i's will be alternately $\lesseqgtr \zeta_m$.

(*) If $\theta = 1$ that is $\eta_1 = 0$, we have complete closure in the phase of the direct blow: which I will call a "sudden closure". In this case, the first equation of the system (9) gives

$$\zeta_1^2 = 1 + 2\rho \qquad \text{(see § 9)}$$

For $\rho = 0{,}5$ we have therefore $\zeta_1 = \sqrt{2}$ and the pressure of the sudden closure is twice the normal.

For $\rho = 1{,}5$ we have $\zeta_1 = \zeta_m = 2$, and the pressure of sudden closure is four times the normal. These two cases of sudden closure are illustrated in fig. 6.

We have therefore an example of a clossing operation of such effect that the first ζ_1^2, ζ_2^2 pressure heights of the total rythme are $< \zeta_m^2$, each of them being, however, greater than the preceding, until one of them becomes larger than the limiting pressure ζ_m, from which instant on the pressure becomes oscillatively asymptotic to ζ_m as in the preceding case.

The law of the pressure in closing presents itself therefore in the form as indicated in the diagram fig. 7 *ter* and the maximum pressure of the total rythme is the one which first exceeds the limiting value ζ_m^2. This maximum pressure can occur in the 2nd, 3rd or i^{th} of the series, in other words the case contemplated is characteristic of the law of pressure of the diverse cases pertaining to the 2nd case, for which it can be stated that the maximum pressure in closure is practically little different from the limiting value ζ_m.

Limiting Cases.

I have remarked that Case 1 always occurs if $\rho < 1$; but it may also occur (that is, we may have $\zeta_1 > \zeta_m$) if ρ is slightly > 1 and the gate operation is executed with sufficient speed. It may be seen from fig. 7 that, putting $\rho > 1$, the less the co-ordinate $-\rho\eta_1$ of C_1, i. e., the more rapid the gate operation, the more the point M is displaced towards the right, so that if ρ is very little > 1, this displacement may be sufficient to make $\zeta_1 > \zeta_m$.

This is examplified in fig. 8, in which, for $\rho = 1{,}12$, the values of ζ_1 and ζ_m are constructed for a gate closure in a time $\theta = 4$, also the values of ζ'_1 and ζ'_m for $\theta = 1{,}15$. This diagram shows by inspection that while $\zeta_1 < \zeta_m$, on the contrary $\zeta_1' > \zeta'_m$ and it is obvious that a velocity of closure can be found for which $\zeta_1 = \zeta_m$.

This limiting case is illustrated by fig. 9, from which it will be seen that the condition for its occurence is that the verticat through C_1 must be equidistant from the segments determing ζ_1, and ζ_m, that is we have

$$\rho = \frac{1}{2}(1 + \zeta_m);$$

and it is obvious that, for this assumption

$$\zeta_1^2 = \zeta_2^2 = \zeta_3^2 = \ldots\ldots \zeta_m^2,$$

i. e., all the pressures of the total rythme will be equal to the limiting value ζ_m^2. This limiting case is evidently the transition from the case in which ζ_1^2 is the maximum pressure of the total rythme and the case in which the maximum pressure is ζ_2^2.

In the same way fig. 10 illustrates the limiting case $\zeta_2 = \zeta_m$ which will occur when the ordinate of C_2 is equal to the mean of ζ_1, and ζ_m, or

$$\rho\eta_1 = \frac{1}{2}(\zeta_1 + \zeta_m)$$

by which condition we have

$$\zeta_2^2 = \zeta_3^2 = \ldots = \zeta_m^2$$

that is all pressures of the total rythme, beginning from the second are equal to the limiting value.

This limiting case is evidently the transition from the case in which ζ_2^2 is the maximum pressure of the total rythme of closure, and the case in which the maximum pressure is ζ_3^2.

In general the ζ^2 of the total rythme are equal to ζ_m^2 beginning at the ith, if the condition is satisfied that

$$\rho\eta_{i-1} = \frac{1}{2}(\zeta_{i-1} + \zeta_m)$$

which is the transition from the case in which ζ_i^2 is the maximum pressure of the total rythme of closure, and the case in which the maximum pressure is ζ'_{i+1}.

Extreme Limiting Cases.

But it is also possible that ρ is large enough and θ small enough so that no pressure of the total rythme will reach the limiting value ζ_m^2, in which case the series of pressures give a growing series of values, the last term of which is a maximum, but nevertheless $< \zeta_m^2$.

In order that such a case should occur it is necessary that all the centers C_i (excepting eventually the last) shall be located as follows:

those with odd indices to the right of the vertical passing through M;

those with even indices above the horizontal passing through M;

and the possibility of such location of the centers C_i results directly from the configuration which the diagram of the interlocked series assumes for large value of ρ and small values of θ. Considering, for instance, fig. 7, it is evident that increasing indefinitely the value of ρ, the center C_4 will be displaced and will be finally located above the horizontal through M, in which case the said condition would be satisfied.

At the end of § 5. I have observed that, in general, pipelines characterized by a large value of ρ (low heads) and by a small value of θ (rapid closure) can be excluded from consideration of normally functioning conduits; they can be, however, hypothetically considered because their study permits of determining those gate operations and speeds of operation which should be avoided, and moreover, pipelines in which the phenomena of constantly increasing pressure occurs are yet within the limits of possibility of technical application, however exceptional they may be.

The reader could, for example, prove by means of a graph similar to those shown, that this case occurs for $\theta = 3$ if $\rho \geq 7$ (for example $y_0 = 20_m$ $v_0 = 3{,}5$ m. per sec., $a = 800_m$, $L = 600_m$., $\tau = 4{,}5$ sec.) which conduit so characterized, while abnormal with respect to the excessive velocity of closure could nevertheless be occasionally met in practice; with the data given the reader will find that $\zeta_3 = 2{,}7$, and that therefore the relative value of the pressure at the instant of closure is $\zeta_3^2 = 7{,}3$ which value accentuates the abnormal conditions of the functioning of the pipe.

With this limiting case all the possible forms of the law of pressure in closing are exhausted.

The single graph of the diagrams of the interlocked series has therefore permitted to establish the general form and some of the important features of the complex phenomena of waterhammer in closure, but its accuracy is insufficient for the pursuit of the systematic generalization of the numerical laws which govern these phenomena; such discussion can only be treated by means of analytical methods and by the use of the cartesian synopsis.

I wish, finally, to call attention to the two practically important results of the preceeding investigation.

1st. — *The maximum relative pressure of the total rythme in closure can be the first of the series* ζ_1^2 (pressure of the direct blow) *or any successive* ζ_i^2, *which have the property of being little different than the limiting value* ζ_m^2.

2nd. — *The values* ζ_1, *and* ζ_m *are obtained by the drawing of a single circle,* γ_1 *of the diagram of the intorlocked series.*

It will be shown in the successive Notes that this same circle furnishes equally the characteristic elements for the opening and alternate operation of the gate.

§ 9. — The pressure of the sudden closure.

Before proceeding with the general analytical discussion of the laws of the pressure in closure due to any duration of the gate operation, we shall briefly illustrate the simple case in which $\theta \lesseqgtr 1$, and the complete closing operation is executed in the phase of the direct blow.

In this case the value ζ^2 of the pressure increases up to the instant of complete closure, corresponding to the value ζ_1, which is obtained * by putting $\eta_1 = 0$ in the 1st of (9)

$$\zeta_1^2 - 1 = 2\rho\,(1 - \eta_1\,\zeta_1); \tag{9}$$

from which is obtained. **

$$\zeta_1^2 = 1 + 2\rho\,. \tag{18}$$

It can be seen that if $\theta < 1$, and the gate remains closed, the pressure will remain at the constant value $1 + 2\rho$ up to the end of the phase, that is, up to the instant $t = \mu$, at which time it enters the regime of oscillating variation, defined by the system of equations (9), putting

$$\eta_2 = \eta_3 = \eta_4 \ldots. = 0,$$

which case will be discussed in a subsequent note.

Equation (18) thus furnishes a second interpretation of the characteristic ρ, which can be defined as the half of the relative surpressure of the *sudden closure*,

This surpressure is therefore smaller the smaller ρ, that is, it is smaller (and therefore theoretically less dangerous) at high than at low heads.

* See also footnote on page 4, and the extremely simple construction (Fig. 6) which gives the value ζ_1.

** From (18) it can obviously be written $Y_1 = (1 + 2\rho)\,y_0 = y_0 + \frac{a\,v_0}{g}$, which expression was alrealy given in the monograph of 1902.

This statement, which at first might seem to be paradoxical, appears, on the contrary, as intuitively logical with reference to the intrinsic significance of the characteristic ρ, which was shown in § 3, to be a number equal to half of the square root of the ratio $\frac{W_0}{W}$ of the kinetic and potential energy of the conduit.

Now, as the normal velocity of flow in the pipe is always within the same limits for low or high heads, the kinetic energy W_0 can be regarded as of the same value for low or high heads, while the potential energy W is the greater, and therefore ρ smaller, the higher the head. It is thus evident that the larger the potential energy W, the smaller will be its relative increase resulting from the absorbtion of all the kinetic energy of the liquid column, which is actually occurring in the case of sudden closure; and finally, the relative surpressure of the sudden closure will be the smaller the smaller is ρ, which is clearly expressed by formula (18).

With the help of the limiting numerical values of ρ (see § 3) it is easy to demonstrate that the fear with which the rapid closure of pipes with high heads is often regarded, and which fear is to an extent justified by the violence of the phenomena which follow the rupture of such pipes, are, nevertheless, without rational foundation.

It can also be seen from (16) that, for a pipeline, the thickness of which is calculated on the basis of a permissible stress equal to 1/4 or 1/5 of the ultimate strength, the limit of rupture can not be reached by the effect of a sudden closure, except if $\rho \gtrsim 1.5$ or 2, while for $\rho < 0.25$ (heads of 200 to 300 m., see fig. 1) the superpressure of the sudden closure does not exceed 50 % of the static head.

After having analysed the phenomena of the resonance we will be able to state that for conduits having a characteristic ρ sufficiently small, it is practically impossible to operate the gate in such a way as to generate waterhammer of sufficient magnitude to rupture the conduit, and only pipes of defective workmanship can burst.

Pipe lines for high heads should therefore always be tested for the pressure of sudden closure, by effecting the gate operation of complete closure in a time somewhat smaller than μ, the duration of the phase. A closure of very short, or praticalty instantaneous duration might, on the contrary, send along the pipe a wave of sudden superpressure, which, propagated to the upper parts which are necessarily less thick, will induce in certain cases dangerous stresses, but. I reserve the discussion of this point for a special note.

§ 10. – The general laws of the pressure in closure.

If the operation of the gate closure has a duration $> \mu$, the study of the laws of the pressure in closing is contained in a systematic analysis of the interlocked series of the values ζ_i . determined by the fundamental system (9)

$$\begin{aligned} \zeta_1^2 - 1 &= 2\rho \quad (1 - \eta_2 \zeta_1) \\ \zeta_1^2 + \zeta_2^2 - 2 &= 2\rho\,(\eta_1 \zeta_1 - \eta_2 \zeta_2) \\ \zeta_2^2 + \zeta_3^2 = 2 &= 2\rho\,(\eta_2 \zeta_2 - \eta_3 \zeta_3) \\ \ldots \quad & \ldots\ldots\ldots \end{aligned} \qquad (9)$$

for diverse values of the characteristic ρ, and assuming that the degrees of opening $\eta_0, \eta_1, \eta_2, \eta_3 \ldots \eta_i$ constitute a linearly decreasing series from $\eta_0 = 1$ to zero.

We have already seen in § 8, how such study can be made partially by the help of the circular diagrams derived from equations (17), and such graphical study has furnished easy and elegant demonstrations of some of the laws of pressure, in a synthetically instructive form.

I am going to undertake now the exhaustive anal tical investigation of the problem with the special view of deriving the cartesian synopsis and diagrams which give a complete illustration of the pressure for all the possible categories of pipelines.

The laws of the pressure in closure are governed by the fundamental principle already partly demonstrated graphically in § 8.

During a closing operation of the gate according to a linear law, all inter locked series of the values ζ_i (which may or may not be of the total rythme) *tend toward a limiting value* ζ_m, *which is independent of the value a of the velocity of propagation.*

If such limiting value exists, it obviously can be determined by the equation obtained in putting

$$\zeta_{i-1} = \zeta_i = \zeta_m$$

in any equation of the system (9) and because

$$\eta_{i-1} - \eta_i = \frac{1}{\theta}$$

the equation which gives the presumed limiting value ζ_m will be (*)

$$\zeta_m^2 - \frac{\rho}{\theta}\zeta_m - 1 = 0 \tag{19}$$

(*) Taking the pressures as unknown, and introducing in equation (19) the notation

$$z = \frac{Y_m}{y_0} = \zeta_m^2$$

it becomes after squaring

$$z^2 - 2z\left(1 + \frac{1}{2}\left(\frac{\rho}{\theta}\right)^2\right) + 1 = 0,$$

which with regard to (20) is nothing else than equation (36) of my first monograph, which. I there obtained through a differential procedure of apparently dubious legitimacy. I arrived at this result by putting the condition

$$\frac{\delta Y}{\delta t} = 0,$$

which is, in a sense, an illegitimate procedure as Y does not vary in a continous way but by a law which shows rythmic discontinuities. The obtained results, however, are not false, because, while the interlocked series $Y_1, Y_2, Y_3 \ldots$ etc., tend to the limit Y_m, the interlocked series $\frac{\delta Y}{\delta t}$ tend toward zero, and the condition $\lim \left(\frac{\delta Y}{\delta t}\right) = 0$ is equivalent to the condition $\lim Y = Y_m$

The fundamental principle which was just given is, however, the only one which gives the true significance and importance of the equations which determine the limiting value of the closing pressure.

from which it appears that $\zeta_m > 1$ and that it is a function of the ratio $\frac{\rho}{\theta}$ only; and, as this ratio

$$\frac{\rho}{\theta} = \frac{av_0}{2gy_0} : \frac{a\tau}{2L} = \frac{Lv_0}{g\tau y_0} \tag{20}$$

it is independent of a which is also true of ζ_m, and which proves the 2nd part of the above statement.

It remains now to prove the first part, i. e., that ζ_m is effectively the limiting value of the interlocked series.

Considering any interlocked series (either of total or of intermediate rythme) and generalizing the statements made in § 8, for the series of total rythme, it will be demonstrated

(A) that the values of the said series may all be inferior to the limiting value given by equation (19), or

(B) that one of the members of the series, for exemple ζ_i, may be greater than ζ_m, in which case the successive numbers ζ_{i+1}, ζ_{i+2}, etc. will be alternately smaller and larger than ζ_m.

Writing equation (19) in the form:

$$\zeta_m^2 + \zeta_m^2 - 2 = 2\rho\,(\eta_{i-1}\,\zeta_m - \eta_i\,\zeta_m)$$

and deducting it from the general equation (9)

$$\zeta^2_{i-1} + \zeta_i^2 - 2 = 2\,\rho\,(\eta_{i-1}\,\zeta_{i-1} - \eta_i\,\zeta_i),$$

we obtain

$$\zeta^2_{i-1} + \zeta_i^2 - 2\,\zeta^2_m = 2\rho\eta_{i-1}\,(\zeta_{i-1} - \zeta_m) - 2\rho\eta_i\,(\zeta_i - \zeta_m) \tag{21}$$

which can also be written in the form

$$\frac{2\rho\eta_{i-1} - \zeta_{i-1} - \zeta_m}{2\,\rho\,\eta_i + \zeta_i + \zeta_m} = \frac{\zeta_m - \zeta_i}{\zeta_m - \zeta_{i-1}}. \qquad (22)_i$$

If, therefore, ζ_{i-1} is yet $< \zeta_m$ and if the numerator of the first member of $(22)_i$ is positive, the second member will also be positive and therefore also $\zeta_i < \zeta_m$.

It follows that if ζ_i is the last term of the interlocked series (that is $\eta_i = 0$), all terms of the series will be smaller than ζ_m.

But, if, on the contrary, ζ_i is not the last member of the series, and if, through the decrease of η, the term $2\rho\eta_{i-1}$ becomes so small that the numerator of the first member of $(22)_i$ will be negative. the same will be true of the second member and therefore necessarily $\zeta_i > \zeta_m$.

I state, moreover that on this assumption the successive terms ζ_{i+1}, ζ_{i+2},... must be alternately $\gtrless \zeta_m$.

To prove this let us write equation $(22)_i$ for the terms of indices i and $i + 1$

$$\frac{2\,\rho\,\eta_i - \zeta_i - \zeta_m}{2\rho\,\eta_{i+1} + \zeta_{i+1} + \zeta_m} = \frac{\zeta_m - \zeta_{i+1}}{\zeta_m - \zeta_i} \qquad (22)_{i+1}$$

and, as the numerator of the first member of $(22)_i$ is negative, the numerator of $(22)_{i+1}$ will be also negative as $\zeta_i > \zeta_m > \zeta_{i-1}$; the second member, therefore, will also be negative, so that, because of $\zeta_i > \zeta_m$ we will have $\zeta_{i+1} < \zeta_m$.

This reasoning is of an entirely general character; the two members of the successive equation $(22)_{i+2}$, $(22)_{i+3}$ etc., are, therefore, always negative and the successive terms of the series are alternately $\gtrless \zeta_m$.

Beginning at the first phase, the broken line of the orthogonal diagram of the pressures ζ^2, therefore, cuts the horizontal $\zeta^2 = \zeta_m^2$ at equal intervals having the value of the phase, at points the ordinates of which constitute an interlocked series of intermediate rythme and having a constant value $\zeta_i^2 = \zeta_m^2$.

This admitted, it will suffice, for demonstration that ζ_m^2 is truly in every case the limiting value of the interlocked series, to prove that:

1. If the values of the terms ζ_i of an interlocked series remain smaller than ζ_m, the following inequalities must stand:

$$\zeta_m - \zeta_1 > \zeta_m - \zeta_2 > \ldots > \zeta_m - \zeta_i$$

2. If a term ζ_i of an interlocked series is $> \zeta_m$, so that the succeding terms are alternately $\gtrless \zeta_m$, the following inequalities must stand

$$\zeta_m - \zeta_{i-1} > \zeta_i - \zeta_m > \zeta_m - \zeta_{i+1} > \text{etc.}.$$

The first statement will be demonstrated by proving that, if the two members of equation $(22)_i$ are positive, they have a value less than 1, that is,

$$2\rho\eta_{i-1} - \zeta_{i-1} - \zeta_m < 2\rho\eta_i + \zeta_i + \zeta_m$$

or

$$\rho(\eta_{i-1} - \eta_i) < \zeta_m + \frac{1}{2}(\zeta_{i-1} + \zeta_i).$$

The truth of this inequality is apparent by observing that on account of equation (19).

$$\rho(\eta_{i-1} - \eta_i) = \frac{\rho}{\theta} = \zeta_m - \frac{1}{\zeta_m}.$$

To demonstrate the second statement, observe, that, given

$$\zeta_{i-1} - \zeta_m < o \qquad \zeta_i - \zeta_m > o,$$

the second member of (21) is evidently negative and so will be the first member from which

$$\zeta^2_{i-1} + \zeta_i^2 - 2\zeta_m^2 < 0$$

$$\zeta_m^2 - \zeta^2_{i-1} > \zeta_i^2 - \zeta_m^2$$

or

$$\zeta_m - \zeta_{i-1} > \zeta_i - \zeta_m$$

The second part of the statement being also demonstrated, we can assume that ζ_m is, in every case, the limiting value of the interlocked series ζ_i.

In conclusion:

The terms of an interlocked series $\zeta_1, \zeta_2 \ldots\ldots \zeta_i$, *can all be* $< \zeta_m$, *or, if one of them is* $> \zeta_m$ *the successive terms are alternately* $\lesseqgtr \zeta_m$.

In order that a term ζ_i *of the series shall be* $> \zeta_m$, *it is necessary that the numerator of equation* $(2^2)_i$ *be negative, that is, that the preceeding term* ζ_{i-1}, *satisfies the condition.*

$$2\rho\eta_{i-1} < \zeta_{i-1} + \zeta_m; \tag{23}$$

if on the contrary, one has

$$2\rho\eta_{i-1} = \zeta_{i-1} + \xi_m, \tag{24}$$

it is evident that for all terms of the interlocked series we would have

$$\zeta_i = \zeta_{i+1} = \zeta_{i+2} = \ldots\ldots = \xi_m.$$

Identical laws control evidently the interlocked series of the pressures $\zeta_1^2, \zeta_2^2 \ldots\ldots \zeta_i^2$, which have as their limits ζ_m^2, as the form of equation of condition (23) and (24) remains naturally the same.

The here stated laws are applicable to all interlocked series whether of the intermediate or of the total rythme. (*)

STANDARD FORMS OF THE LAWS OF THE PRESSURES IN CLOSURE (Fig. 11 to 17).

Applying the preeceding results to the series of the pressures of the total rythme due to closure, it will be possible to establish systematically the characteristic standard forms of the pressure and consequently to represent, by means of orthogonal diagrams, the several forms of the pressure law which result from the conditions (23) and (24).

Putting successively $i = 1, 2, 3, 4$, etc. into these equations, we will consider them in this order.

First case, $i = 1$.

From (23) .... $2\rho < 1 + \zeta_m$,

where ζ_m must be expressed as a function of ρ and θ by means of equation (19).

If this condition is satisfied. the first pressure of the total rythme which surpasses the limiting value ζ_m^2 is the pressure of the direct waterhammer ζ_1^2; the general form of the pressure curve is then represented by the diagram fig. 11, and the pressure of the direct blow ζ_1^2 is found to be the maximum pressure of the series of values of total rythme; it is, moreover, excepting certain special cases which will be discussed later, (§ 13), the absolute maximum of the pressure in closure.

This is the same case which has already been studied in § 8 and which is represented by figures 4, 4*bis*, 4*ter* and 5.

(*) Equations (23) and (24) were established in § 8, based upon simple graphical considerations which are susceptible to serve as a starting point of a real graphical theory of the water hammer; the discussion and use of these equations, however, can only he done by the analytical way; hey can, moreover, be graphically interpreted only by means of the cartesian synopsis.

Taking now the limit of equation (24) $2\rho = 1 + \zeta_m$

we have $\zeta_1^2 = \zeta_m^2$ and also $\zeta_1^2 = \zeta_2^2 = \zeta_3^2 = \ldots\ldots \zeta_m^2$;

the form of the pressure curve in closure is therefore that represented by fig. 12; the pressure reaches a limiting value at the end of the first phase and remains almost exactly at this value throughout the remainder of the closing operation.

This case is evidently a transition to the

Second case, $i = 2$.

From (23) we have: $2\rho\eta_1 < \zeta_1 + \zeta_m$,

where ζ_1 and ζ_m can be expressed as functions of ρ and θ by help of the first equation of the system (9) and equation (19). The pressure of the total rythme which first surpasses the limiting value ζ_m^2 is the first pressure of the counterblow ζ_2^2; this pressure is at the same time the maximum pressure of the series, and the form of the pressure curve in closure is represented in fig. 13.

Putting, as a limiting case $2\rho\eta_1 = \zeta_1 + \zeta^m$,

the pressure will grow during the first two phases; it will reach the limiting value ζ_m^2 at the and of the second phase and will approximately retain this value during the remainder of the closing operation, as shown in fig. 14. (see also the circular diagram fig. 10).

Third case, $i = 3$

From (23): $2\rho\eta_2 < \zeta_2 + \zeta_m$,

in which ζ_2 can be expressed in terme of ρ and θ by the first two equation of (9) and ζ_m by equation (19).

The pressure of the total rythme which first surpasses the limit ζ_m^2 is, in this ease, ζ_3^2; this is the maximum pressure of the series and the form of the pressure curve is represented in fig. (15).

In the limiting case $2\rho\eta_2 = \zeta_2 + \zeta_m$,

the pressure will grow during the first three phases up to the limiting value ζ_m^2 which will be approximately constant during the balance of the closing operation.

Etc. etc.

It is evidently superfluous to continue this demonstration of the results which are similar throughout.

Extreme limiting case.

But if the conditions (23) and (24) are not reached for any of the pressures of the total rythme during the closing operation, the series of closing pressures will continuosly grow; the pressure can however, in the last phase, and at an instant of intermediate rythme, reach or surpass the limiting value ζ_m^2 as shown in fig. (16), or not reach it at all, as indicated in fig. (17).

The analytical investigation therefore confirms and completes the statements made in § 8; the reader has, in the fig. 11 to 17 the standard form of all possible laws of the pressure curve in closure.

The extraordinary variety of form of these standards show why the attempts made to express these laws by a single equation must necessary remain futile.

The logical study of the conditions under which any of these standard forms is realized, reduces itself, therefore, to the study of the equations which result in each particular case, from the application of the conditions (23) and (24); I will deduce from this study, with the help of the cartesian synopsis described in § 5, the criterions which will serve as a basis to the classification of conduits from the point of view of the pressures of total rythme due to closure.

§ 11. — The Cartesian Synopsis of classification of conduits, from the point of view of the maximum pressure of the total rythme, due to a closing operation.

From consideration of the relations (23) and (24), noting that

$$\eta_{i-1} = 1 - \frac{i-1}{\theta}$$

while ζ_{i-1} and ζ_m, from equation (9) and (19), are functions of ρ and θ, it evidently results

A) that the relation (23)

$$2\,\rho\,\eta_{i-1} < \zeta_{i-1} + \zeta_m$$

into which successively $i = 1, 2, 3$, etc. are introduced, generate a series of inequalities between the functions ρ and θ, each of which, interpreted in the cartesian synopsis (see § 5) determine a zone Σ_i which contains the double (quintuple) infinity of the conduits for which the law of closure has the same form as those of the cases illustrated in fig, 11, 13, and 15; this form, moreover, depends on the value attributed to i, as we have seen it in the preceding paragraph.

B) that the relation (24)

$$2\,\rho\eta_{i-1} = \zeta_{i-1} + \zeta_m, \tag{24}$$

in which successively $i = 1, 2, 3$, etc. are introduced, generates a series of equations between ρ and θ, which, interpreted in the plan of the cartesian synopsis determine the set of the lines s_i which limit the zones Σ_i; each of these lines is the locus of a single (quadruple) infinity of the conduits, for which the law of pressure in closure has the form defined by one of the diagrams fig. 12 and 14, in which the pressure reaches the limiting value in an instant of total rythme and remains sensibly constant at this value until the end of the gate operation.

The plat of the cartesian synopsis (see fig. 18), which we will now study, will give the reader a first concrete example of the use it can be put in the investigation and systematic representation of the laws of the waterhammer phenomena.

It will be useful to remember, at this junction, the fundamental system (9)

$$\begin{aligned} \zeta_1^2 - 1 &= 2\rho\,(1 - \eta_1\,\zeta_1) \\ \zeta_1^2 + \zeta_2^2 - 2 &= 2\rho\,(\eta_1\,\zeta_1 - \eta_2\,\zeta_2) \qquad (9) \\ &\cdots\cdots\cdots \\ \zeta^2_{i-1} + \zeta_i^2 - 2 &= 2\rho\,(\eta_{i-1}\,\zeta_{i-1} - \eta_i\,\zeta_i), \\ &\cdots\cdots\cdots \end{aligned}$$

which represent the interlocked series ζ_i of the total rythme, where

$$\eta_1 = 1 - \frac{1}{\theta}, \qquad \eta_2 = 1 - \frac{2}{\theta}, \ldots \eta_i = 1 - \frac{i}{\theta},$$

and also equation (19)

$$\zeta_m^2 - \frac{\rho}{\theta}\,\zeta_m - 1 = 0 \qquad (19)$$

which gives the limiting value toward which the series converge.

GENERAL PROPERTIES OF THE LOCI s_i.

The loci s_i are curves which possess, in the positive quadrant of the synopsis, a branch of hyperbolic form having asymptotes parallel to the axes, i. e.:

1st The loci s_i have a common vertical asymptote being the line

$$\rho = 1;$$

2nd Each locus s_i has as its horizontal asymptote the line

$$\theta = i - 0{,}5.$$

The first of these postulates can be demonstrated by putting $\theta = \infty$ (which is equivalent of the slowing down of the operation indefinitely) and evidently obtaining

$$\text{Lim.}\ \eta_{i-1} = 1 \qquad \text{Lim.}\ \zeta_{i-1} = 1 \qquad \text{Lim.}\ \zeta_m = 1.$$

which values, introduced in (24) give as the equation of the common asymptote of all all loci s_i, $\rho = 1$, which demonstrates the truth of postulate 1.

To demonstrate postulate 2, observe that from (19)

$$\frac{\zeta_m}{\rho} = \sqrt{\frac{1}{4\theta^2} + \frac{1}{\rho}} + \frac{1}{2\theta}$$

from which, for $\rho = \infty$:

$$\text{Lim.}\ \frac{\zeta_m}{\rho} = \frac{1}{\theta}$$

in the same way, from the system (9), we obtain, after dividing by ρ^2 and putting $\rho = \infty$, the system:

$$\text{Lim.}\left(\frac{\zeta_1}{\rho}\right)^2 = \qquad -2\eta_1\,\text{Lim.}\left(\frac{\zeta_1}{\rho}\right),$$

$$\text{Lim.}\left(\frac{\zeta_1}{\rho}\right)^2 + \text{Lim.}\left(\frac{\zeta_2}{\rho}\right)^2 = 2\eta_1\,\text{Lim.}\left(\frac{\zeta_1}{\rho}\right) - 2\eta_2\,\text{Lim.}\left(\frac{\zeta_2}{\rho}\right),$$

etc. etc.

which necessarily result in

$$\text{Lim.}\frac{\zeta_1}{\rho} = \text{Lim.}\frac{\zeta_2}{\rho} = \ldots\ldots = \text{Lim.}\frac{\zeta_i}{\rho} = 0.$$

Dividing equation (24) by ρ, putting $\rho = \infty$, and introducing the values obtained for $\frac{\zeta_m}{\rho}$ and $\frac{\zeta_{i-1}}{\rho}$, we get

$$2\eta_{i-1} = 2\left(1 - \frac{i-1}{\theta}\right) = \frac{1}{\theta},$$

or, reducing
$$\theta = i - 0{,}5\,; \tag{25}$$

which is the general equation of the horizontal asymptote of the loci s_i, which had to be demonstrated.

This demonstration may seem somewhat hazy to the reader who has not yet familiarized himself with the use of the synopsis; but this is a first typical example of the simplicity, elegance and generality of this method. The accuracy of the preceding postulates, moreover, will be shown by the study of the curves s_i, which I now will undertake (*)

THE ZONE Σ_1 AND THE LOCUS s_1.

If, in equation (23) we put $i = 1$, as already observed in the preceding § there is obtained the relation:

$$2\rho < 1 + \zeta_m \tag{26}$$

which, after eliminpting ζ_m by means of equation (19) and after some reduction gives

$$\rho < \frac{4\theta - 1}{4\theta - 2},$$

(*) Only that portion of the curves s_i situated in the positive quadrant $(+\rho, +\theta)$ will be studied here; the branches of these curves situated in the quadrant $(+\rho, -\theta)$ also interest us as they find their application in the theory of the waterhammer of opening; the change of the sign of θ is in fact equivalent to the change of the direction of the gate operation. I only note here that, in the quadrant $+\rho, -\theta$, the curves s_i constitute a set of hyperbolic curves contained between $\rho = 0$ and $\rho = 1$ as a common asymptote; for more details, see Note III (which is the study of the opening operation) and the corresponding cartesian synopsis.

which characterizes those conduits, the pressure in closure of which has the form represented by fig. 11. This condition, as a rule, is always satisfied when $\rho \leqq 1$ as already demonstrated by means of the circular diagram fig. 4 (§ 8); on the other hand, if $\rho > 1$ the relation includes the two cases ($\zeta_i \gtrless \zeta_m$) represented by the circular diagram fig. 8.

The zone Σ_1, embracing the conduits characterized by the relation (26), is limited, in the field of the cartesian synopsis, by the curve s_1, fig. 18, defined by the equation

$$\rho = \frac{4\theta - 1}{4\theta - 2}, \tag{27}$$

which is the equation of an equilateral hyperbola having for asymptotes the lines

$$\rho = 1 \quad \text{and} \quad \theta = 0{,}5,$$

already derived from the general equation (25), and the branch of which situated in the positive quadrant intersects the horizontal $\theta = 1$ at point S_1, having an absissa $\rho = 1{,}5$.

This branch was drawn, in fig. 18, along the vertical asymptote and continued to point S_1 only, that part which would be located above $\theta = 1$ has no technical meaning, because, if $\theta = 1$ it is a case of sudden closure, which phenomena is discussed in § 9 to which the reader is referred.

The zone Σ_1, therefore, is located between the vertical axis $\rho = 0$, the horizontal, $\theta = 1$ for a distance of 1.5, and the portion drawn of the hyperbolic branch s_1, a curve which is the locus representing the conduits for which $\zeta_1 = \zeta_2 = \zeta_3 \ldots = \zeta_m$, and where the pressure of closing has the form of the diagram fig. 12.

This zone Σ_1, which embraces the conduits for which the maximum pressure of the total rytme due to closure is the pressure of the dlrect blow ζ_1^2, from a technical point of view is the most interesting zone of the synopsis, first, because it covers the conduits of high and very high heads (which can be seen by inspecting the diagram superimposed to the synopsis in fig. 18), and, moreover, because it has very important properties for other kinds of gate operations, which properties will be discussed later.

THE ZONE Σ_2 AND THE LOCUS s_2.

If, in equation (24) we put $i = 2$, there is obtained, as already mentioned in the preceding paragraph:

$$2\rho\eta_2 = \zeta_1 + \zeta_m; \tag{28}$$

This is the equation of the curve s_2 which bounds the zone Σ_2 adjacent to zone Σ_1; this new zone Σ_2 embraces the conduits for which ζ_2^2 is the maximum pressure of the total rytme due to a closing operation, and for which the pressure has the form of diagram Fig. 13; the curve s_2 is the locus of conduits for which $\zeta_2 = \zeta_3 \ldots = \zeta_m$ and for which the pressure has the form of diagram Fig. 14.

Eliminating in (28) ζ_1 and ζ_m by means of the first of equations (9) and by equation (19); also, putting

$$\eta_1 = 1 - \frac{1}{\theta}$$

and multiplying by 2θ, we obtain

$$\rho(6\theta - 7) = 2\sqrt{\rho^2(\theta-1)^2 + (2\rho+1)\theta^2} + \sqrt{\rho^2 + 4\theta^2} \qquad (29)$$

This is the equation of s_2 in terms of ρ and θ, which is a quadratic in ρ and is, as yet, convenient enough for the determination of the curve s_2 . .

This curve, in the positive quadrant of the synopsis, possesses a branch of hyperbolic form, having as orthogonal asymptotes $\rho = 1$ and $\theta = 1.5$, conforming to equation (25); it passes, moreover through the following points:

$\theta =$ 2	3	4	5	6	7	10	20
$\rho =$ 4.04	1.99	1.59	1.43	1.33	1.27	1.18	1.08

which permitted to draw the curve in Fig. 18; this branch runs along the vertical asymptote and stops at point S_2 ($\theta = 2$, $\rho = 4.04$); it was not drawn further, above $\theta = 2$ as we assumed a gate operation executed in a time $\theta \geq 2$.

The zone Σ_2 therefore is bounded by the curves s_1 and s_2, the horizontal $\theta = 2$, which gives it a form of a curved triangle with one apex at infinity.

THE ZONE Σ_3 AND THE LOCUS s_3.

Finally, let us put $i = 3$ in equation (24) and we obtain

$$2\rho\eta_2 = \zeta_3 + \zeta_m ; \qquad (30)$$

this is the equation of the curve s_3 which, according to the relation (25), has the orthogonal asymptotes $\rho = 1$ and $\theta = 2.5$; this curve s_3 bounds, together with s_2 and the orizontal $\theta = 3$, (fig. 18), the zone Σ_3 which embraces the conduits for which the maximum pressure of the total rythme in closure is ζ_3^2; the curve s_3 is the locus of the conduits for which

$$\zeta_3 = \zeta_4 = \ldots = \zeta_m .$$

Eliminating in equation (30) ζ_3 by means of the first and second equation of the system (9) and ζ_m by equation (19), we obtain a very complicated expression in ρ and θ which it is superfluous to quote here, and which can only be solved by successive approximations; the reader may prove that the series of the following points satisfy this equation with sufficient accuracy:

$\theta =$ 3	4	5	10
$\rho =$ 7.00	2.75	2.01	1.33

by which coordinates curve s_3 was drawn in fig. 18.

It is evident that the determination of the coordinates of the points located on curves s_4, s_5 etc., becomes more and more difficult and cumbersome.

LIMITING FORM EQUATION OF s_i.

The determination of the curves s_i by such a difficult procedure, however, has neither theoretical value nor practical value because it does not tell anything about the general properties of the curves s_i, and has no practical importance, because, as already stated (see the circular diagrams of the interlocked series), for $i > 3$, the value ζ_i differs from ζ_m by a quantity which can be considered negligible with respect to the errors which are made (even with the greatest care) in the estimation of the constants which define the conduit. A few numerical examples will convince the reader of this fact.

However, for the sole reason of not leaving incomplete the theoretical discussion of this question. I will try to establish some laws of the curves s_i for the whole field of the synopsis.

Let us observe, for this purpose, that if ζ_{i-1} differs only very little from ζ_m, equation (24)

$$2\rho\eta_{i-1} = \zeta_{i-1} + \zeta_m \tag{24}$$

also differs very little from equation

$$\rho\eta_{i-1} = \zeta_m, \tag{31}$$

therefore, we can consider this last equation as being the limiting form toward which tends the general equation (24) of the loci s_i, when i increases.

Introducing, in equation (31)

$$\eta_{i-1} = 1 - \frac{i-1}{\theta} \qquad \zeta_m = \sqrt{\left(\frac{\rho}{2\theta}\right)^2 + 1} + \frac{\rho}{2\theta}$$

we obtain, after some easy transformation

$$\rho = \frac{\theta}{\sqrt{(\theta - i)^2 + \theta - i}}, \tag{32}$$

which is the limiting form of the equation of the loci s_i in terms of θ and ρ.

The curves represented by this equation differ very little from the curves s_i in the region of their vertical branches which are asymptotes to $\rho = 1$; on the other hand, they are always below these curves, in the region of the horizontal branches because the asymptotes of the horizontal branches of the curves s_i are given by $\theta = i - 0.5$, while the asymptotes of the limiting curves, (32), are given by $\theta = i$. Applying, for instance, equation (32) to the case $i = 2$ and comparing curve s_2 with its limiting form, we obtain

θ =	2	2,5	3	4	5	6	7	10
ρ (from29) =	4.033	2.488	1.991	1.595	1.426	1.333	1.272	1.178
ρ (from 32) =	∞	2.899	2.122	1.633	1.443	1.342	1 278	1.178

These figures demonstrate that while $i = 2$ is less than the limit beginning with which we propose to use equation (32), a close approximation is already reached for $\theta = 5$ and for $\theta = 10$ the coincidence of the two sets of figures extends to the 3d decimal.

As the limiting equation (32) gives the greater approximation the greater i, its application for the drawing of curves $s_3, s_4 \ldots\ldots s_i$ will appear justified, provided however, that the horizontal branches are raised to make them asymptotic to $\theta = i - 0.5$ and not to $\theta = i$. Fig. 18 was completed in this manner.

THE LOCUS $S_1 S_2 S_3 \ldots . S_i$.

It is clear that the conduits represented by the point S_i, where the loci s_i intersect the horizontals corresponding to $\theta = i$ enjoy the property that the pressure of closure reaches the limiting value ζ_m^2 (for the total rythme) at the instant of complete closure.

In fact, for a conduit located on s_i, this limiting value of the pressure is reached at the end of the $i.^{th}$ phase; and this instant coincides precisely with that of the complete closure when the conduit is also located at $\theta = i$.

I state, moreover, that the points $S_1 S_2 \ldots S_i$ are located on a curve which is the locus of conduits for which the pressure reaches the limiting value ζ_m^2 at the very instant (of intermediate rytme) of the complete closure.

The existence of such a locus can be derived, by reason of continuity, from the examination of the laws of the pressure of closure, for a series of conduits characterized by increasing values of ρ.

Let us consider, for example, the series of conduits the speed of operation of which is defined by a value of θ, $3 < \theta < 4$, and by increasing values of ρ; these conduits evidently are represented in the synopsis by the points of a horizontal line which, from left to right, successively cuts the three zones Σ_1, Σ_2, Σ_3, in which are located the conduits for which the pressure of closure surpasses the limiting value ζ_m^2 at an instant of the first, second and third phases.

Now, we must admit, by reason of continuity, that at the right of the zone Σ_3 there must be located the conduits for which the pressure surpasses the limiting value at an instant of a fraction of the phase (incomplete 4th phase) preceding the closure, as represented by fig. 16.

But, we know, that on the same line, more to the right, i.e., beyond a certain value of ρ, there exist conduits for which the pressure remains constantly inferior to the limiting value, as represented by fig. 17; it is therefore evident that on this line a point must be located which represents a conduit for which the pressure reaches its limiting value at the very instant of complete closure.

As, on the other hand, the locus of conduits satisfying this condition must evidently contain the points $S_1 S_2 \ldots S_i$, it must have the form of the nearly straight line indicated by dotting in fig. 18.

I shall give here, in order to demonstrate the application of the method, the analytical determination of this locus, for the branch $S_1 S_2$, i.e. for $1 < \theta < 2$ only.

Denoting by an index' the intermediate values of η and ζ (i. e. these corresponding to non-integer values of θ) and making the conditions

$$\zeta_2' = \zeta_m \qquad \eta_2' = 0$$

and

$$\eta_1' = \frac{1}{\theta}$$

the two first equations of (9) will give

$$\zeta_1'^2 - 1 = 2\rho\left(1 - \frac{\zeta'_1}{\theta}\right)$$

$$\zeta_1'^2 + \zeta_m^2 - 2 = 2\rho\frac{\zeta'_1}{\theta}.$$

Eliminating ζ_1' and substituting ζ_m vith help of equation (19). we have

$$8\sqrt{\rho^2 + (2\rho+1)\theta^2} - \sqrt{\rho^2 + 4\theta^2} = 9\rho + 4\theta^2 \qquad (33)$$

which is the equation of the sought locus. It is easy to demonstrate that this locus passes trough the points S_1 and S_2.

With $\theta = 1$, equation (33) will be satisfied by the sole value $\rho = 1.5$, which is the absissa of S_1.

Putting $\theta = 2$, equation (33) will be satisfied by the value (*) $\rho = 4.038$ which is the absissa of S_2.

The investigation of the segments $S_2 S_3$, $S_3 S_4$ etc. (**) results in more and more complicated analytical expressions, which, however: do not add anything new to the conclusions stated.

THE SYNOPSIS OF CLASSIFICATION OF CONDUITS FROM THE POINT OF VIEW OF THE PRESSURES IN CLOSURE (fig. 18).

The set of the loci s_i and the locus $S_1\ S_2 \ldots S_i$ furnish, so to speak, the general skeleton of the synopsis of classification of conduits from the point of view of pressures in closure; this synopsis is represented by the graph fig. 18; its description can be recapitulated as followe:

1st. The line $\theta = 1$ bounds, in the upper portion of the synopsis, the zone of sudden closure which I shall designate by Θ_1; it comprises those conduits for which the pressure increases up to the instant of complete closure and reaches the value $1 + 2\rho$.

2nd. The curves $s_1, s_2, s_3 \ldots s_i$ bound the zones $\Sigma_1, \Sigma_2, \Sigma_3, \ldots \Sigma_i$, each of which comprise the conduits for which the pressure surpasses the limiting value ζ_m^2 at an instant which belongs respectively to the first phase for Σ_1, to the second for Σ_2, to the third for Σ_3 to the i^{th} for Σ_i. The pressure of the total rythme which immediately follows this instant is then the maximum pressure of the series of the pressures of the total rythme.

(*) In reality, equation (33), for $\theta = 2$, becomes an equation of the 3rd degree which has 2 equal roots

$$\rho = \frac{1}{6}(11 + 5\sqrt{7}) = 4.038,$$

the absissa of point S_2, and a third root

$$\rho = \frac{6}{7},$$

the absissa of point $S_{1,2}$ where curve s_1 cuts the line $\theta = 2$. This point therefore is an isolated point of equation (33); the conduit which it represents, satisfies, in fact, the condictions which determine equation (33).

(**) By which is shown that the equation for $S_2 S_3$ is satisfied by the isolated points $S_{1,2}$ and $S_{2,3}$, also that for $S_3 S_4$ is satisfied by points $S_{2,3}$ and $S_{3,4}$, etc. See fig. 18.

Each of the curves s_i is the locus of conduits for which the pressures of the total rythme are equal to the limiting value $\zeta_m{}^2$ beginning with the *i*th.

3rd. Between the horizontal pointed portions of the zone Σ_3, Σ_2 . . . Σ_i are included the zones which I have designated by Θ_2, Θ_3 . . . Θ_i comprising the conduits for which the pressure, growing constantly during the closing operation, reaches and surpasses the limiting value $\zeta_m{}^2$ at an instant which forms part of the last incomplete phase preceeding the closure.

4th. The zones Θ_2, Θ_3 . . . Θ_i are limited by the line S_1 S_2 S_3 . . . S_i which line is the locus of conduits for which the pressure, growing constantly during the closing operation, reaches the limiting value $\zeta_m{}^2$ at the very instant of complete closure; to the right of this line there is an unlimited angular zone designated by Θ which comprises the conduits for which the pressure grows constantly during the closing operation but never reaches the limiting value $\zeta_m{}^2$.

As a synthesis of what precedes, I can evidently state as follows:

Disregarding zone Θ_1 of the sudden closure, all zones of the synopsis of classification can be grouped in two principal regions; to wit:

α) The region constituted by the ensemble of zones Σ_1, Σ_2 . . . Σ_i; this region comprises the conduits for which the maximum pressure of total rythme, due to a closing operation, occurs at one of intermediate instants of total rythme of the operation; this maximum pressure is $> \zeta_m{}^2$ for the conduits located in the zone $\Sigma_{i,j}$; it is $= \zeta_m{}^2$, for the conduits located upon the loci s_i which separate the zones.

The absolute maximum pressure (always $> \zeta_m{}^2$) occurs therefore at a well determined instant of the closing operation.

β) The region constituted by the aggregate of the zones Θ, Θ_2, Θ_3, . . . Θ_i; this region comprises the conduits for which the maximum pressure of the total rythme due to closure, occurs at the last instant of the total rythme which preceeds the closing; this maximum pressure is always $> \zeta_m{}^2$.

The absolute maximum pressure therefore always occurs at the end of the gate operation; it is $< \zeta_m{}^2$ for the zone Θ and $> \zeta_m{}^2$ for the zones Θ_2 Θ_3 . . . Θ_i, while it is $= \zeta_m{}^2$ for the conduits situated on the line S_1 S_2 S_3, . . . S_i which separates the zone Θ from the zones Θ_2, Θ_3 . . . Θ_i.

These results, it seems, fully justify the title « Synopsis of Classification of conduits from the point of view of pressures in closure » which I adopted for this cartesian diagram of the curves s_i, because it classifies in a logical manner all possible conduits with reference to the law of pressure variation during a closing operation.

Let us yet investigate what is the relative importance of the several zones of the synopsis from the point of view of the probability of actual occurance of conduits which they represent. Let us observe first, that the line S_1 S_2 S_3 . . . S_i which limits the zone Θ also bounds, with a certain approximation (and disregarding a few exceptional cases) those conduits which are practically possible and those which are not, because the conduits located on this line must either be exceptionally long or their gates be operated with extraordinary speed. Let us consider for instance, the 4 conduits represented by the points.

		S_1	S_2	S_3	S_4
for which	$\theta =$	1	2	3	4
and	$\rho =$	1,5	4	7	10

If, for example, these conduits were operated in very short times, such as $\tau = 3, 4, 5$ seconds, they had to have the following lengths, which are easy to find with the help of the characteristic diagram.

Conduit		S_1	S_2	S_3	S_4	
	$y_0 =$	100	20	15	10	m.
(see diagram)	$v_0 =$	3.5	3.3	3	3	m/sec
(fig. 1):	$a =$	850	750	705	650	m/soc
if $\tau = 3$ sec	$L = \frac{a\tau}{2\theta} =$	1275	565	350	245	m.
if $\tau = 4$ sec	$L = \quad =$	1700	750	465	325	m.
if $\tau = 5$ sec	$L = \quad =$	2125	940	585	405	m.

It can be stated that these lengths are in effect out of proportion with the corresponding heads so that we can exctude as impractical, at least as conduits for water-power, all those situated in the zone Θ, in other words all conduits for which the maximum pressure in closure is $< \zeta_m^2$. The only conduits which remain for close investigation are therefore, as a rule, those which are located in the zones Θ_i and Σ_i, for which the limiting pressure ζ_m^2 is always reached and overtopped during the closing operation of the gate.

But, as I have stated already, excepting the conduits of zone Σ_1, and perhaps of Σ_2, this limiting pressure is surpassed by so little for all the rest of the field of the synopsis that it can by justly considered as being the maximum value of the pressure in closure.

This last remark strongly accentuates the importance of the limiting value ζ_m^2 for this category of problems and for almost the whole of the field of the synopsis; but it should not be lost ot view that it is not applicable to a very interesting category of conduits, that of high heads ($\rho < 1$).

§ 12. — The Synopsis as a Cartesian diagram of the Pressure ζ_m^2 and ζ_1^2 of closure.

I have pointed out, at the and of § 5, the fact that the method of synopsis lends itself extremely well to the construction of diagrams giving the numerical values of magnitudes characterizing the phenomenon of the water hammer, that is, the pressures; it is sufficient, for the purpose, to draw, in the synoptic plan, the loci of conduits for whic these pressures attain predetermined relative values; as a first application, I will construe the diagrams of the pressures of closure.

DIAGRAMS OF THE LIMITING PRESSURES ζ_m^2 (see fig. 19).

The observatinns at the and of the preeceding paragraph called attention to the overwhelming tmportance which must be attributed to the limiting pressure ζ_m^2 in almost the whole of the synoptic plan. It, therefore, seems opportunate to try to establish, in the first instance, the diagram of ζ_m^2, especially as the character of this diagram is extremely simple.

In fact, the plat of the loci of conduits, for which the limiting (relative) pressure ζ_m^2 attains determined values, in nothing else but a set of straight lines passing through the origin.

From equation (19) which gives the limiting value of ζ_m:

$$\zeta^2_m - \frac{\rho}{\theta} \zeta_m - 1 = 0, \qquad (19)$$

it follows, that the plat of the loci of the conduits for which ζ_m^2 reaches given values is defined by

$$\frac{\rho}{\theta} = \zeta_m - \zeta_m^{-1}, \qquad (34)$$

which equation justifies the preceeding statement

By meens of this equation (34) it is easy to obtain the folloving system of values, for a series of values of ζ_m^2 increasing from unity:

ζ^2_m	$\frac{\rho}{\theta}$	ζ^2_m	$\frac{\rho}{\theta}$	ζ^2_m	$\frac{\rho}{\theta}$
= 1,0	= 0	= 2,0	= 0,707	= 4,0	= 1,500
1,1	0,095	2,2	0,809	4,5	1,650
1,2	0,183	2,4	0,904	5,0	1,783
1,3	0,263	2,6	0,992	6,0	2,041
1,4	0,338	2,8	1,076	7,0	2,268
1,5	0,408	3,0	1,155	8,0	2,475
1,6	0,474	3,2	1,230	9,0	2,667
1,7	0,537	3,4	1,302	10,0	2,846
1,8	0,596	3,6	1,370	12,0	3,175
1,9	0,653	3,8	1,436	15,0	3,615
				etc.	etc.

by means of which values the rays of straight lines passing through the origin have been platted the plat giving the diagram of the limiting pressures ζ_m^2 (Fig. 19).

In the same diagram are also platted the curves s_i which bound the zones Σ_i and Θ so that on the diagram there can be found, at the same time, the limiting pressnre ζ_m^2 (which can be considered as tbe maximum pressure for the whole of the field of the synopsis, excepting the regions Θ, Σ_1 and part of Σ_2) and also the instant of the operation at which this pressure occurs.

For example, for the conduits represented by the points.

1st $\rho = 2, \theta = 4$, the conduit is situated in Σ_3 and the pressure will reach the relative value 1.65 at about the middle of the 3rd phase, or at 5/8 of the operation.

2nd $\rho = 3, \theta = 7.5$, the pressure will have the relative value 1.5 near the middle of the 6th phase, or at 11/15 of the operation, etc.

This diagram makes it possible to immediatoly recognise, whether or not an incomplete operation can raise the pressure to the limit which it would reach if the operation be continued.

In the fig. (19), the straight lines representing $\zeta_m^2 =$ const., were shown dotted in the zones $\Theta, \Theta_1, \Sigma_1$ and in a portion of Σ_2, i. e., at the zones where the maxima of the pressure in closure differ sensibly from ζ_m^2.

DIAGRAM OF THE PRESSURES ζ_1^2 (see Fig. 20).

Disregarding completely the zone Θ which, as already mentioned in the preceeding paragraph does not present any features of technical interest, let us investigate the zones Θ_1 and Σ_1, in which the maximum pressure of total rythme in closure is always the pressure ζ_1^2 (pressure of sudden closure for the zone Θ_1, and direct blow for the zone Σ_1) determined by the first equation of the system (9).

Zone Θ_1. If $\theta \leqq 1$ as explained at length in § 9, the pressure, at the instant of complete closure reaches the value

$$\zeta_1^2 = 1 + 2\rho . \qquad (18)$$

which equation is obtained by putting $\eta = 0$ in the first equation of the system (9); it retains that value (which we have called the pressure of sudden closure) up to the end of the first phase.

Because the value of ζ_1^2, as given by equation (18) is indipendent of θ, the loci of the conduits for which the pressure of sudden closure reaches determined values will be represented by the series of vertical segments between $\theta = 0$ and $\theta = 1$, and characterized by the absissae.

$$\rho = \frac{1}{2}(\zeta_1^2 - 1).$$

These loci were platted in Fig. 20 and it is believed that the plat does not need further explanation.

It will be noted that for $\rho = 1.5$, we have $\zeta_4^2 = 4$; the loci $\zeta_4^2 = 4$ and $\zeta_m^2 = 4$ meet at point S_1, the intersecfion of s_1 and $\theta = 1$.

Zone Σ_1.

This zone comprises those conduits for which the mayimum pressure of the total rytme in closure is the pressure of direct blow ζ_1^2 determined by the first equation of the system (9).

$$\zeta_1^2 - 1 = 2\rho\,(1 - \eta_1\zeta_1) . \qquad 1^{st} \text{ of } (9)$$

in which is put

$$\eta_1 = 1 - \frac{1}{\theta} .$$

By means of this substitution, the general equation of the loci of these contuits, for which ζ_1^2, renders given values, becomes, as can be easily proven

$$\rho = \frac{\zeta_1^2 - 1}{2} \cdot \frac{\theta}{\zeta_1 - (\zeta_1 - 1)\,\theta} . \qquad (35)$$

If a series of constant values is attributed to ζ_1^2, this equation represents a series of equilateral hyperbolae passing through the origin and having respectively the asymptotes

$$\rho = -\frac{\zeta_1 + 1}{2}; \qquad \theta = +\frac{\zeta_1}{\zeta_1 - 1} .$$

Those branches of these hyperbolas which lie in the synoptic quadrant must naturally meet the homologuous straight lines of the series $\zeta_m = \text{const.}$, in points of the locus s_1 (see Fig. 20) because this locus exactly corresponds to the conditions $\zeta_1 = \zeta_m$.

This locus s_1 therefore can also be defined as being the locus of the points of intersections, of the homologous elements of the series (34) and (35).

Fig. (20) therefore constitutes a diagram of the pressures of the direct blow ζ_1^2, which permits us to determine by inspection, for the conduits situated to the left of s_1 (zone Σ_1) how and by what amount ζ_1^2 is $> \zeta_m^2$, and, conversely, for the conduits to the right of s_1, how and by what amount ζ_1^2 is $< \zeta_m^2$.

By an analogous process it is possible to construct the loci of the conduits for which the first pressure of the counter blow ζ_2^2 has a predetermined value, (*) which is the maximum of the pressures of the total rythme in closure for the conduits situated in Σ_2.

That part of the locus $\zeta_2^2 = \text{const.}$, located in the zone Σ_2 is thus a curve convex downward which passes through the two points where $\zeta_m^2 = \text{const}$, s_1 and s_2. The plat obtained by this laborius method, however, has no real interest partly because the curves deviate very little from the straight lines $\zeta_m^2 =$ const., in the lower portion of Σ_2, and partly because the intermediate maxima, which will be treated in the next paragraph, have a much larger importance thant the pressures of total rythme in the upper part of Σ_2 and in Θ_2 i. e. (fig. 18) in the region located to the right of s_1 and below $\theta = 1$, up to about $\theta = 4$.

The same remark applies to the loci $\zeta_3^2 = \text{const.}$, $\zeta_4^2 = \text{const.}$, etc., the plat of which is even more difficult, but which can be considered, for all practical purposes, as merging with the straight lines $\zeta_m^2 = \text{const.}$, for the regions located in the respective zones Σ_3, Σ_4, etc.

§ 13. — The Synopsis of classification of the conduits from the point of view the maximum pressure of the intermediate rythme in closure.

It was clearly pointed out in section 11, that for the conduits located in the limiting zone Θ and within the horizontal zones $\Theta_1, \Theta_2, \Theta_3$ etc., situated between the pointed portions of the zones Σ_i, the maximum pressure occurs at the instant of complete closure, both for the total or intermediate rythme.

I now propose to demonstrate that, to the contrary, for the conduits located in the field of the zones Σ_i, there are produced, in closing, maxima of the intermediate rythme $\zeta_1^2, \zeta_2^2 ..... \zeta_i^2$, which are greater than the maxima of the total rythme on the basis of which latter we have established the subdivisions of the regions of the synoptic field occupied by the ensemble of these zones Σ_i; we will now establish the lews of these intermediate maxima.

Although this study should evidently be the sequence of those made in preceeding paragraphs, neverthelese an equal or even greater importance must be attributed to it, as this study has the result of completely exhausting the problems of the maximum pressures in closure.

The reader will find without difficulty that this study, in general, reduces to that of the interlocked series of the values

$$\frac{\delta \zeta_i}{\delta t} \text{ and } \frac{\delta^2 \zeta_i}{\delta t^2},$$

and that it can be developed without the help of those demonstrated in § 10 and § 11; at the same time, it can give place to synoptic representation, which, from a technical point of view, bring the problem to a closer solution than the synopsis of classification (Fig. 18).

(*) The method to be used consists in eliminating ζ_1, between the first and second equation of (9), and in putting $\zeta_2 = \text{const.}$; but the resulting equation in ρ and Θ can only be solvet by trial.

I have, nevertheless, preferred to start with the exposition of results which follow a partial study of the problem, based on the investigation of finite differences which exist between the terms $\zeta_1, \zeta_2, \zeta_3 \ldots\ldots \zeta_i$ of the interlocked series of the total rythme, and this for two reasons.

First, the periodic discontinuties which occur in the variation of the pressure have made it easy to grasp its general laws; it was sufficient to concentrate out attention upon the laws according to which vary the salient points of the broken line representing the pressure in closure; the fact that the interlocked series $\zeta_1, \zeta_2 \ldots\ldots \zeta_i$ tend rapidly toward a limiting value ζ_m, moreover, give a great pratical value to the results of this investigation and to the therefore derived synoptic representation.

Second, this propriety permitted us to plat what could be called the skeleton of the diagram of the pressure for the entire duration of the operation, while the study of the differential quotients is necessarily limited, due to the very fact of periodic discontinuity, to the duration of each phase of the phenomenon, so that no conclusions can be derived from same which would hold beyond the phase considered. If gives us, substantially, the laws according to which the curvature of the pressure line varies between two instants of the total rythme and indicates the eventual presence of maxima, the numerical value of which, however, must be calculated by means of the fundamental equations.

I. – GENERAL FORMULAE.

For convenience, let us denote by an index ′, the symbols relative to the values of the intermediate rythme; differentiating with respect to t the fundamental equations and considering the conditions:

$$\eta'_i = 1 - \frac{t}{\theta}\,; \qquad \frac{\delta \eta'_i}{\delta t} = -\frac{1}{\theta}\,,$$

we obtain (see also equations (12) of § 4):

$$(\zeta'_1 + \rho\eta'_1)\frac{\delta\zeta'_1}{\delta t} = \frac{\rho}{\theta}\,\zeta'_1$$

$$(\zeta'_1 - \rho\eta'_1)\frac{\delta\zeta'_1}{\delta t} + (\zeta'_2 + \rho\eta'_2)\frac{\delta\zeta'_2}{\delta t} = \frac{\rho}{\theta}\,(\zeta'_2 - \zeta'_1) \qquad (36)$$

$$(\zeta'_2 - \rho\eta'_2)\frac{\delta\zeta'_2}{\delta t} + (\zeta'_3 + \rho\eta'_3)\frac{\delta\zeta'_3}{\delta t} = \frac{\rho}{\theta}\,(\zeta'_3 - \zeta'_2)$$

etc. etc.

which system of equations, together with the equation (9), connect the series of interlocked values:

$$\zeta'_1,\ \frac{\delta\zeta'_1}{\delta t},\quad \zeta'_2,\ \frac{\delta\zeta'_2}{\delta t},\ \ldots\quad \zeta'_i,\ \frac{\delta\zeta'_i}{\delta t},$$

with regard to the instants which differ between them by the interval μ of a phase.

From equations (36) follow:

$$\frac{\delta\zeta'_1}{\delta t}=\frac{\rho}{\theta}\cdot\frac{1}{\zeta'_1+\rho\eta'_1}\zeta_1$$
$$\frac{\delta\zeta'_2}{\delta t}=\frac{\rho}{\theta}\cdot\frac{1}{\zeta'_2+\rho\eta'_2}\cdot\left[\zeta'_2-\frac{2}{\zeta'_1+\rho\eta'_1}\zeta'^2_1\right] \qquad (36\ bis)$$
$$\frac{\delta\zeta'_3}{\rho t}=\frac{\rho}{\theta}\cdot\frac{1}{\zeta'_3+\rho\eta'_3}\cdot\left[\zeta'_3-\frac{2}{\zeta'_2+\rho\eta'_2}\cdot\left(\zeta'^2_2-\frac{\zeta'_2-\rho\eta'_2}{\zeta'_1+\rho\eta'_1}\zeta'^2_1\right)\right]$$
$$\text{etc.} \qquad\qquad \text{etc.}$$

which equations permit of the discovery of the existence of intermediate maxima of the pressure in the several phases, except, however, in the first phase, (the phase of the direct blow) in which the pressure is continuously increasing as demonstrated by the first equation of the system (36 *bis*).

We will make a complete study only for the second phase and will append some information concerning the 3rd and 4th phases, altough it should be noted that, beginning with the 3rd phase, the problem does not present any practically interesting features. We know, in fact, that in this phase the pressure in closure approaches so close the limiting pressure ζ^2_m that all investigation regarding the exact value of the actual pressure becomes of purely academic interest.

II. — INTERMEDIATE MAXIMUM JN THE 2nd PHASE.

The conditions for the existence of an intermediate maximum of: ζ'_2 are:

$$\frac{\delta\zeta'_2}{\delta t}=0 \qquad \frac{\delta^2\zeta'_2}{\delta t^2}<0; \qquad (37)$$

remembering that the second of these conditions means that the line of pressure must be convex upward (because, in the contrary case we would have a minimum instead of a maximum).

We shall nov demonstrate that this condition is always satisfied when the first condition holds, in other words; there may be a maximum in the second phase, but never a minimum.

Differentiating, for this purpose, the first equation of the system (36) and also the equation obtained in adding each member of the first and second equation of the same system we obtain:

$$(\zeta'_1+\rho\eta'_1)\frac{\delta^2\zeta'_1}{\delta t^2}=\frac{\delta\zeta'_1}{\delta t}\left(\frac{2\rho}{\theta}-\frac{\delta\zeta'_1}{\delta t}\right) \qquad (38)$$
$$2\left(\frac{\delta\zeta_1}{\delta t}\right)^2+2\zeta'_1\frac{\delta^2\zeta'_1}{\delta t^2}+(\zeta'_2+\rho\eta'_2)\frac{\delta^2\zeta'_2}{\delta t^2}=\frac{\delta\zeta'_2}{\delta t}\left(\frac{2\rho}{\theta}-\frac{\delta\zeta'_2}{\delta t}\right)$$

The first equation of the system (36) permits us to conclude without difficulty, that the second member of the first equation of the system (38) is always positive, and that, consequently $\frac{\delta^2\zeta'_1}{\delta t^2}$ is also constantly positive, so that the pressure curve, during the first phase (the phase of direct blow) is always concave upward. Now, because the first and second terms of the first member of the second equation of the system (38) are necessarily positive quantities, and because the second member or that equation vanishes for the point corresponding to the maximum value of the curve, it follows that

$$\frac{\delta^2 \zeta'_2}{\delta t^2} < 0$$

which had to be demonstrated.

The condition that a maximum should occur during the second phase is therefore always obtained in putting the second equation of (36 *bis*) equal to zero, which gives

$$\zeta'_2 = \frac{2\zeta'^2_1}{\zeta_1 + \rho\eta'_1}. \qquad (39)$$

This equation is that of the plat of loci of the conduits which possess the indicated properties.

If we wish, in fact, that the maximum should occur at an instant $t = m$ of the second phase (m being a value between 1 and 2), we must put:

$$\eta'_2 = 1 - \frac{m}{\theta}, \qquad \eta'_1 = 1 - \frac{m-1}{\theta}, \qquad (40)$$

and, by means of the first and second equation of (9), eliminate ζ'_1 and ζ'_2 from equalion (39); we will obtain a curve ρ_{2m} of the synopsis in terme of ρ and θ, an this curve will be the locus of the conduits for which a maximum pressure in closure will occur at an instant $t = m$ of the second phase.

If we attribute to m a series of values included between 1 and 2, equation (39) will give the plat of the loci in question. These loci have a common vertical asymptote, $\rho = 1$, but have no horizontal asymptote which distinguishes them from the loci s_i.

Putting, in fact, $\theta = \infty$: $Lim\, \eta'_1 = 1$; $Lim\, \zeta'_1 = Lim\, \zeta'_2 = 1$ equation (39) will give

$$1 = \frac{2}{1+\rho}, \qquad \text{soit: } \rho = 1;$$

on the other hand, if we transform (39) and vrite:

$$\frac{\zeta'_2}{\rho}\left(\frac{\zeta'_1}{\rho} + \eta'_1\right) = 2\left(\frac{\zeta'_1}{\rho}\right)^2,$$

and remembering, as already remarked in Section 11, that for $\rho = \infty$, $Lim\, (\zeta_i : \rho) = o$, it will be clearly seen that the two members vanish independently.

The plat of the loci considered is evidently contained between the limiting loci obtained in putting $m = 1$ and $m = 2$ in equation (39): these limiting loci are those of conduits for which the maximum pressure in closure occurs respectively at the beginning and at the end of the second phase.

Limiting locus $\sigma_{2\cdot 1}$ (Fig. 21 and 22).

Applying equation (39) for the case $m = 1$, and putting, for this purpose:

$$\eta'_1 = \eta_0 = 1; \qquad \zeta'_1 = \zeta_0 = 1; \qquad \zeta'_2 = \zeta_1;$$

we obtain:

$$\zeta_1 = \frac{2}{1+\rho},$$

Eliminating ζ_1 by means of the first equation of (9), where:

$$\eta_1 = 1 - \frac{1}{\theta},$$

is substituted, and reducing, we get:

$$\theta = \frac{4\rho(\rho+1)}{3-\rho^2(2\rho+1)}; \tag{41}$$

which is the equation of the locus sought.

This locus passes through the points

$\rho = 0.45$	0.50	0.6	0.7	0.8	0.9	1.0
$\theta = 1.00$	1.20	1.74	2.61	4.31	9.35	∞

and is platted in Fig. 21 and 22.

This locus therefore is situated to the left of the asymptote $\rho = 1$, in the zone Σ_1. It is limited, on the other hand, on account of the physical phenomenon which it represents, by the line $\theta = 1$, as the closing operation which induces this phenomenon must last at least to the beginning of the second phase ($\theta \gtreqless 1$).

Limiting locus $\sigma_{2\cdot 2}$

Applying equation (39) for $m = 2$, and putting

$$\eta'_1 = \eta_1 = 1 - \frac{1}{\theta}; \qquad \zeta'_1 = \zeta_2; \qquad \zeta'_2 = \zeta_2$$

which is the same as dropping the indices and considering the symboles as relating to the instants of the total rythme with indices 1 and 2 respectively.

The limiting locus will become

$$\zeta_2 = \frac{2\zeta_1^2}{\zeta_1 + \rho\eta_1}, \tag{42}$$

in which equation ζ_1 and ζ_2 must be replaced by their values given in the first and second equations of (9), where there has been put

$$\eta_1 = 1 - \frac{1}{\theta}, \qquad \eta_2 = 1 - \frac{2}{\theta};$$

This method results in very complicated analytical expressions, and it is preferable to proceed by successive approximations and to determine the pairs of values ζ_1 and ζ_2 which satisfy equation (42).

The sought locus will pas through the points

$\rho = 2$	2.5	3	4	5	7	∞
$\theta = 2.07$	1.83	1.67	1.46	1.36	1.26	1.00

This locus therefore will pass to the right of the asymptote $\rho = 1$, in the zone Σ_2; it is limited, in view of the physical phenomenon represented, by the line $\theta = 2$, as the gate operation resulting in the phenomenon must last at least to the end of the second phase ($\theta \gtreqless 2$).

Limiting locus $\sigma_{2\cdot 0}$

Because the limiting locus $\sigma_{2\cdot 1}$ stops at $\theta = 1$ and the locus $\sigma_{2\cdot 2}$ at $\theta = 2$, we can say that a certain locus $\sigma_{2.m}$ (where m is contained between 1 and 2) must stop at $\theta = m$, at a point which represents a conduit for which the algebraic maximum of the pressure occurs at the very instant of complete closure.

The locus of these extreme points, which we will designate by $\sigma_{2\cdot\theta}$ is naturally determined by the same general equation (39).

$$\zeta'_2 = \frac{2\zeta_1'^2}{\zeta'_1 + \rho\eta'_1}, \qquad (39)$$

and by the supplementary condition $m = \theta$, where θ is contained between 1 and 2, which, by means of equation (40) gives

$$\eta'_2 = o; \qquad \eta'_1 = \frac{1}{\theta}.$$

On this assumption, the first and second equations of the system (9) furnish

$$\zeta'_2 = \sqrt{4\rho\eta'_1\zeta'_1 - 2\rho + 1}, \qquad (43)$$

from which the sought locus can be derived by equations (39) and (43) and eliminating ζ'_1, by means of the first equation of (9). This locus must necessarily pass through the extreme points of $\sigma_{2\cdot 1}$ (for $\theta = 1$) and of $\sigma_{2\cdot 2}$ (for $\theta = 2$).

It is, however, more simple to determine, from equations (39) and (40), by successive approximations a series of points of the locus $\sigma_{2\cdot\theta}$ for a series of values of θ, comprised between 1 and 2.

$\theta = 1$	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
$\rho = 0.45$	0.54	0.65	0.78	0.91	1.06	1.23	1.42	1.63	1.84	2.07

which made it possible to plot the locus $\sigma_{2\cdot\theta}$ in Fig. 21.

The points of the preceding table, considered respectively as the extreme points of $\sigma_{2;1\cdot 1}$; $\sigma_{2;1\cdot 2}$; $\sigma_{2;1\cdot 3}$... make it possible to draw with reasonable accuracy the plot of these loci, the general character of which is indicated by similarity with that of the limiting loci $\sigma_{2\cdot 1}$, and $\sigma_{2\cdot 2}$ and by the common asymtote, $\rho = 1$ (Fig. 21).

The plat of the $\sigma_{2\cdot m}$ drawn in Fig. 21 therefore constitutes a diagram of classification from the point of view of the maximum pressure in closure, of the conduits situated in the zone which we will designate by $\Sigma_{1\cdot 2}$ (because it is at boundry of the zones Σ_1 and Σ_2) limited by the loci $\sigma_{2\cdot\theta}$, $\sigma_{2\cdot 1}$ and $\sigma_{2\cdot 2}$.

These conduits are characteristized by the fact that the algebraic maximum of the pressure in closure occurs in the second phase; we talk here of algebraic maximum, as the numerical maximum also occurs during the second phase, exactly at the instant of the complete closure, for all conduits situated between $\theta = 1$ and $\theta = 2$ and to the right of $\sigma_{2\cdot\theta}$.

In order to demonstrate better the conclusions derived from this study, we have represented graphically (See Fig. 21 bis and 21 ter) the laws of the, pressure in closure for 5 different conduits A_2, B_2, C_2, D_2, E_2, located at $\theta = 1.5$ and for four conduits A_3, B_3, C_3, D_3, located at $\theta = 2.5$; these diagrams are, we believe self explanatory.

Finally in order to gain a positive idea of the numerical value of the intermediate maxima in relation to the value of the pressures of the total rythme, I calculated these maxima for four conduits located on s_1, for which the pressures of the total rythme are equal to ζ_m^2, and which have a law of pressure variation in closure, the form of which is illustrated by the diagram Fig. 12.

Altough the curve s_1, does not belong to the plat of the $\sigma_{2\cdot m}$, it is very little different from $\sigma_{2,1\cdot 6}$, so that for the conduits located on s_1, and in the zone $\Sigma_{1\cdot 2}$, the maximum pressure cccurs at the instant which corresponds to the 6/10 of the second phase, while for the conduits located on the upper portion of s_1, between $\sigma_{2\cdot 0}$ and $\theta = 1$, the maximum pressure occurs at the instant of closure.

We obtain

Conduits defined by the Co-ordinates of the Points of s_1		Pressure of the total rythme	Maximum Pressure
$\theta = 1.25$	$\rho = 1.333$	$\zeta_m^2 = 2.78$	$\zeta_2'^2 = 3.13$
$= 1.50$	$= 1.25$	$= 2.25$	$= 2.55$
$= 2.00$	$= 1.167$	$= 1.78$	$= 1.80$
$= 3.00$	$= 1.10$	$= 1.44$	$= 1.47$

from which values it may be seen that, already for $\theta = 3$, the maximum of the intermediate rythme differs only very little from the pressure of the total rythme; theen researches therefore become interesting for the conduits characterized by the small values of θ (rapid closure).

III. — INTERMEDIATE MAXIMA IN THE 3RD PHASE.

The conditions of the intermediate maxima in the third phase are:

$$\frac{\delta \zeta'_3}{\delta t} = o \qquad \frac{\delta^2 \zeta'^2_3}{\delta t^2} < o, \tag{44}$$

but these two conditions are not interconnected as were the conditions (37) of the preceeding case, for which it was stated that the second is always satisfied when the first one stands. The first of the conditions (44) can, in fact, correspond to a minimum which may occur at certain regions of the synoptical field; to visualize this it is sufficient to inspect diagram 12; it is easy to see that such a region must exist around the lucus s_1; Fig. 14 on the contrary, will show us that the region in which the first of the conditions (44) corresponds to a maximum, must occur around s_2.

The preceeding can be demonstrated analytically by means of conveniently developing the system of the relations (38); such demonstration is, however, of a questionable utility; it is sufficient, for our purposes, that we have established that the zone in which the first of the conditions (44) corresponds to a maximum, and the limits of which we now will try to find, extends around s_2.

If the condition of maximum stated previously is introduced in the 3rd equation of the system (36) we find.

$$\zeta'_3 = \frac{2}{\zeta'_2 + \rho \eta'_2}\left(\zeta_2'^2 - \frac{\zeta_2' - \rho \eta'_2}{\zeta_1' - \rho \eta'_1}\zeta_1'^2\right), \tag{45}$$

which, with the three first equations of the system (9) in which is to be put

$$\eta'_3 = 1 - \frac{m}{\theta}; \qquad \eta'_2 = 1 - \frac{m-1}{\theta}; \qquad \eta'_1 = 1 - \frac{m-2}{\theta}, \tag{46}$$

will determine the plat of the loci σ_{3m} of the conduits for which the pressure of closure presents a maximum at the instant $t = m$ of the third phase (m being a value betwen 2 and 3).

Limiting locus $\sigma_{3 \cdot 2}$.

Let $m = 2$, and consequently

$$\eta'_2 = 1 - \frac{2}{\theta}; \quad \eta'_3 = 1 - \frac{1}{\theta}; \quad \eta_{/1} = \eta_0 = 1,$$

$$\zeta'_3 = \zeta_2; \quad \zeta'_2 = \zeta_1; \quad \zeta'_1 = \zeta_0 = 1,$$

equation (45) will give, for the limiting curve $\sigma_{3 \cdot 2}$, the locus of conduits fo which the algebraic maximum of the pressure occurs at the beginning of the 3rd phase.

$$\zeta_2 = \frac{2}{\zeta_1 + \rho\eta_1}\left(\zeta_1^2 - \frac{\zeta_1 - \rho\eta_1}{1 + \rho}\right); \tag{47}$$

eliminating from this ζ_1 and ζ_2, by means of equations 1 and 2 of the system (9) the following pairs of points can be derived through which pass the locus sought.

$\theta = 2$	3	4	5	7	∞
$\rho = 1{,}80$	1,47	1,34	1,21	1,19	1,00

by means of which this locus was platted in Figures 21 and 22.

Due to the character of the phenomenon represented, this locus must naturally be limited by $\theta = 2$ (because $\theta \leqq 2$).

Limiting locus $\sigma_{3 \cdot 3}$.

Putting $m = 3$, and also

$$\eta'_3 = 1 - \frac{3}{\theta}; \quad \eta'_2 = 1 - \frac{2}{\theta}; \quad \eta'_1 = 1 - \frac{1}{\theta}.$$

and dropping the indices in equation (45), we will obtain the equation of the locus $\sigma_{3 \cdot 3}$ of the conduits for which the algebraic maximum of the pressure occurs at the and of the third phase. This locus, naturally limited by the line $\theta = 3$ passes through the points

$\theta = 3$	4	5	7	10	∞
$\rho = 3.22$	2.24	1.87	1.50	1.25	1.00

through which it was platted in Figs. 21 and 22.

Limiting locus $\sigma_{3 \cdot \theta}$.

Putting, finally, $m = \theta$ (where $2 < \theta < 3$) and

$$\eta'_3 = o; \quad \eta'_2 = \frac{1}{\theta}; \quad \eta'_1 = \frac{2}{\theta},$$

the relation (45) will furnish the equation of the locus of the extreme points of the curves $\sigma_{3 \cdot m}$, which locus represents the conduits for which the algebraic maximum of the pressure occurs at the very instant of closure.

The loci $\sigma_{3 \cdot \theta}$, $\sigma_{3 \cdot 2}$ and $\sigma_{3 \cdot 3}$ define the zone $\Sigma_{2 \cdot 3}$ (which extends at the same time over Σ_2 and Σ_3), comprising the conduits for which the maximum pressure in closure occurs during the 3rd phase; we can draw, in this zone, by means of the known points of $\sigma_{3 \cdot \theta}$, the loci $\sigma_{3 \cdot m}$ (dotted in Fig. 21) for which the maximum pressures occur at a given instant of the third phase.

For the conduits located to the right of $\sigma_{3\cdot 0}$ and between $\theta = 2$ and $\theta = 3$, the absolute numerical maximum pressure occurs exactly at the instant of closure.

IV. — INTERMEDIATE MAXIMUM IN THE 4TH PHASE

The conditions of the intermadiate maxmum in the 4th phase will be

$$\frac{\delta \zeta'_4}{\delta t} = o; \qquad \frac{\delta^2 \zeta'_4}{\delta t^2} < o,$$

to which the same observations can be applied as made with respect to the conditions (45); these condition will permit to determine, point by point, by means of equation 4 of the system (36 bis) the limiting loci $\sigma_{4\cdot 3}$, $\sigma_{4\cdot 4}$, $\sigma_{4\cdot 0}$, which bound the zone $\Sigma_{3,4}$, comprising the conduits for which the maximum pressure in closure occurs during the 4th phase.

The locus $\sigma_{4\cdot 3}$, bounded evidently by the straight line $\theta = 3$ passes through the points

$\theta = 3$	4	5	7	∞
$\rho = 3$	2.10	1.80	1.45	1.10

and the locus $\sigma_{4\cdot 4}$, bounded by the line $\theta = 4$ passes through the points:

$\theta = 4$	5	7	10	∞
$\rho = 4.40$	2.72	1.95	1.50	1.00

while the locus $\sigma_{4\cdot 0}$, with a small concavity upward, connects the extreme points of the two preceeding loci, as shown in Fig. 22.

By the same procedure we could determine the zones $\Sigma_{4\cdot 5}$, $\Sigma_{5\cdot 6}$, etc. relating to the fifth, sixth, etc. phases.

Fig. 22 therefore represents the synopsis of classifications from the point of view of a maximum pressure of intermediate rythme in closure.

The zones $\Sigma_{1\cdot 2}$, $\Sigma_{2\cdot 3}$, $\Sigma_{3\cdot 4}$, have the form of a curved triangle, one of the apexes of which is in infinity on the line $\rho = 1$, and which partly overlap (*); they comprise the conduits for which the maximum pressure of the intermediate rythme in closure, occur respectively in the second, third, fourth, etc., phases. Their median lines are the loci $s_1, s_3, s_5 .. s_i$ and they are limitied, in their upper parts, by arcs of finite lengths $\sigma_{2\cdot 0}$, $\sigma_{3\cdot 0}$... Conduits located to the right of these arcs reach the numerical absolute maximum pressure at the instant of closure. This synoptical representation, as can be seen, differs from the conception of the limiting pressure $\zeta^2{}_m$, upon which was based the synopsis of classification of conduits, from the point of view of the pressures of the total rythme, and represented by Fig. 18.

However, these two synopses are mutually complementary; they are connected by the fact that the curves $s_1, s_3 ... s_i$, which in the one case bound the zones Σ_i, in the other case are the median lines of the zones $\Sigma_{1\cdot 2}$, $\Sigma_{2\cdot 3}$, etc. It is a remarkable result that the condition of maximum applied in one instance to a finite series of the values of the pressure corresponding to successive phases, and in the other instance to the infinite series of the values of the pressure within each phase, conducts, in both cases, to diagrams defined by loci which all have te line $\rho = 1$ as their common assymptote.

(*) This particular overlap signified that the conduits represented by points common to two zones have two maxima close to each other, in the phases of the order i and $i + 1$.

§ 14. — The Synopsis

General Diagram of the pressure maximums in closure.

The results of the preceeding Section enable us to proceed to the construction of a general diagram representing the pressure in closure.

ZONE $\Sigma_{1\cdot 1}$

Let us so designate that part of zone Σ_1, comprised between the axis θ and $\sigma_{2\cdot 1}$ which separates it from the zone where the maxima of the intermediate rythme occur in the second phase (Fig. 22).

In this zone $\Sigma_{1\cdot 1}$, the maximum pressure is always that of the direct blow, ζ_1^2, the diagram Fig. (20): discussed in Section 12, is therefore valid in this zone

The locus of the conduits for which the maximum pressure ζ_1^2, reaches a given value is therefore the hyperbole represented by equation (35); the plat of these hyperboles gives the diagram Fig. 23, of the maximum pressures for the regions located in the zone considered.

This zone contains exclusively conduits of very bigh heads (see also the diagram of the characteristic ρ); it has a very great practical importance, especially in the upper portion, because, in the case of necessarily very long conduits, and of considerable duration of phase, it is generally difficult to have very large values of θ (relatively slow closures).

ZONE $\Sigma_{1\cdot 2}$, $\Sigma_{2\cdot 3}$ etc.

In the zone $\Sigma_{1\cdot 2}$, there are shown the loci $\sigma_{2\cdot m}$ of the conduits, for which the maximum pressure of the intermediate rythme at the instant $t = m$ of the second phase. We can, therefore, determine, for a series of points (or conduits) of each of these loci, the numerical value of the maximum pressure occurring, and, by interpolation, we can plot the loci of the conduits for which the maximum pressure reaches a given value.

We can proceed in the same manner for the loci $\sigma_{3\cdot m}$ of the zone $\Sigma_{2\cdot 3}$; but this has no particular interest except for the upper portion of this zone, because, as already demonstrated, these intermediate maximums differ only very little from the limiting pressure ζ_m^2 when θ is greater than 3 or 4.

The loci $\zeta_m^2 =$ const. were so determined and plotted in Fig. (23). These loci constitute a partial diagram of the maximum pressures in closure, applicable to high and very high heads.

THE REGION TO THE RIGHT OF THE CURVES $\sigma_{i.\theta}$.

We have previously observed that, in the zones

$$
\begin{aligned}
1 < \theta < 2, &\text{ to the right of } \sigma_{2.\theta} \\
2 < \theta < 3, &\quad \text{»} \quad \sigma_{3.\theta} \\
i - 1 < \theta < i, &\quad \text{»} \quad \sigma_{i.\theta}
\end{aligned}
$$

the maximum pressure is always the pressure $\zeta^2_{i,\theta}$ which occurs at the instant of complete closure; by means of systematically performed computations and convenient interpolations, it is easy, therefore, to construct within each of these zones, the loci of conduits for which the maximum pressure reaches a given value.

These loci, along the line $\theta = 1$, meet the vertical segments corresponding, to the diagram of sudden closure, and join, on $S_1\, S_2\, S_3 \ldots S_i$ (see the synopsis. Fig. (18) the radial lines issuing from the origin, which represent the plat of the limiting pressures ζ_m^2.

In § 11 we have, in fact, remarked that, for the conduits located on $S_1\, S_2\, S_3 \ldots S_i$, the pressure in closure is constantly increasing and reaches the value ζ_m^2 at the instant of complete closure.

Finally, it may be of interest to note that for the segments of the loci in question, which are situated in the region $1 < \theta < 2$, it is possible to find a general expression of the equation in terms of ρ and θ, which makes the platting much easier.

Applying, for this purpose, the first and second equation of (9) to the instants of the intermediate rythme and putting the conditions

$$\eta'_2 = o\,; \quad \eta'_1 = \frac{1}{\theta}\,; \quad \zeta'_2 = \zeta_{2\cdot\theta}\,,$$

we have

$$\zeta'^2_1 - 1 = 2\,\rho\left(1 - \frac{\zeta'_1}{\theta}\right),$$

$$\zeta'^2_1 + \zeta^2_{2\,\theta} \quad 2 = 2\,\rho\,\frac{\zeta'_1}{\theta}\,;$$

from which

$$4\,\rho\,\frac{\zeta_1'}{\theta} - 2\,\rho + 1 = \zeta^2_{2\cdot\theta}\,.$$

Assigning a series of numerical values > 2 to the pressure $\zeta_2^2{}_{\cdot\theta}$ of the instant of closure, and eliminating ζ'_1 by means of the first of the two preceding equations, we obtain, after some reductions:

$$\theta = \frac{2\,\rho\,\sqrt{4\,(\rho + 1) - 2\,(\zeta^2_{2\cdot\theta} - 1)}}{2\,\rho + \zeta^2_{2\cdot\theta} - 1}\,, \tag{48}$$

by the help of which we can plat the loci in question for any desired value of $\zeta_2^2{}_{\cdot\theta}$ in the zone $1 < \theta < 2$, located to the right of $\sigma_{2\,\theta}$.

It is easily verified, that, for $\theta = 1$, equation (48) results in $\zeta_2^2{}_{\cdot\theta} = 1 + 2\,\rho =$ the pressure of sudden closure; on the other hand, if we compute the value of ρ for $\theta = 1{,}5$, and $\theta = 2$, and for a given series of integer value of $\zeta_2^2{}_{\cdot\theta}$, we obtain

	$\zeta^2_{2\cdot\theta} = 3$	4	5	6	7
$\theta = 1$	$\rho = 1$	1,5	2	2,5	3
$= 1{,}5$	$= 1{,}53$	2,13	2,70	3,26	3,81
$= 2$	$= 2{,}15$	2,84	3,49	4,10	4,70

which figures demonstrate that these loci are lines of slight curvature only; the abscissa of the middle point, on the line $\theta = 1.5$, differs, in fact, only by about 0,04 from the abscissa of the middle point of a straight line connecting the extreme points situated on $\theta = 1$ and $\theta = 2$.

In the same manner, but only by means of easily performed successive approximations, can be determined the loci $\zeta^2_{3\cdot\theta} =$ const., which constitute the prolongation of the former in the zone $2 < \theta < 3$, located to the right of $\sigma_3 \cdot \theta$, and in a general way, the loci $\zeta^2{}_{i.\theta} =$ const., in zone $i-1 < \theta < i$, to the right of $\sigma_{i.\theta}$; it is, however, superfluous, from a practical point of view, to pursue this research beyond $\theta = 3$ or 4.

GENERAL DIAGRAM.

Finally, if we combine in a single synoptical diagram, the partial diagrams of the different zones the study of which we just finished, *i.e.*,

A) for the zone $0 < \theta < 1$, the diagram of the pressures of sudden closure of Fig. 20

B) for the zone $1 < \theta < 2$, to the right of $\sigma_2 \cdot \theta$;
for the zone $2 < \theta < 3$, to the right of $\sigma_3 \cdot \theta$;

. . . .

for the zone $i < \theta < i+1$, to the right of $\sigma_{i+1 \cdot \theta}$, the diagrams, just described, of the loci $\zeta^2{}_{i.\theta} =$ const.;

C) in the zone $\Sigma_{1 \cdot 1}$, the diagram of the pressures of the direct blow, the hyperbola segments of Fig. 20;

D) in the zones $\Sigma_{1 \cdot 2}, \Sigma_{2 \cdot 3} \ldots . \Sigma_{i-1, i}$ the diagram of Fig. 23 conveniently extended to the upper parts of the zones, (however, for values i and θ less than 3 or 4), and, for the balance of these zones, the diagram of the limiting pressures $\zeta_m{}^2$ of Fig. 19, the corresponding loci being connected by smooth curves, we obtain the general diagram of the maximum pressures in closure (Fig. 24) which is a condensed resume of the results of all the researches described in this Note.

The partial diagram (Fig. 23) which gives the larger scale detail of the region comprising the conduits of high heads, constitutes a complement of the general diagram, the value of which will be appreciated in application to actual cases. In the general diagram (Fig. 24) are also platted the loci $s_1\ s_2 \ldots s_i$, which serve to determine the instant of the gate operation, when the pressure reaches its maximum; this element of the problem, in many cases, has a great pratical value.

In order to illustrate the use of the diagram by a numerical example, we will assume a conduit characterized by $\rho = 4.5, \theta = 20$, From fig. 24 it can be seen that this conduit falls on s_{16} and therefore the maximum pressure, $\zeta_m{}^2 = 1{,}25$, is reached after 16 rythmes, in other words at 4/5 of the closing operation; such a conduit may correspond to the following data:

$$y_0 = 20\ ; \quad v_0 = 2{,}5\ ; \quad a = 700\ ; \quad L = 140\ ; \quad \tau = 8 \text{ sec.}\ ;$$

In this case, the maximum pressure would be reached at the instant $t = 6{,}4$ sec; if the closing operation of the gate would be incomplete and should stop at a time < 6.4, sec. the maximum pressure indicated by the diagram could not be reached.

NOTE III.

THE WATERHAMMER OF OPENING

Introductory observations.

In the relation

$$\eta = 1 - \frac{t}{\theta},$$

which defines a closing operation executed according to a linear law, the symbol θ designates the time necessary to complete the closure of the orifice of outflow; this same symbol will denote, per contra, the time necessary to double the initial area of the orifice, when we deal with an opening operation also executed in accordance with a linear law defined by

$$\eta = 1 + \frac{t}{\theta},$$

However, these definitions have a precise meaning only if the closing or opening operations which they define are started from a state of regime corresponding to a given part of the area of the orifice, to which part I attributed the value of unity ($\eta_0 = 1$).

Should the opening operation start at zero orifice area, the preceding definitions of the notations ρ and θ would be faulty.

As a matter of fact, for all conduits, the value of ρ is equal zero at a regimen characterized by $v_0 = 0$; and because $\psi_0 = 0$; i. e., $\eta = \infty$, we have, in all cases, $\theta = 0$, which is meaningless. For this case the fundamental formulas must be conveniently modified.

Therefore, I shall discuss separately the two cases:

(A.) Opening operations of conduits in service, i e., of conduits for which ψ_0 and v_0 are greater than zero; this study evidently presents a close correlation in all its details with that made in the preceding Note on the waterhammer in closure.

(B.) Opening operations for the placing in servtce of conduits in which the water is at rest, i. e., of conduits, the initial condition of which is defined by $\psi_0 = 0$ and $v_0 = 0$; as just mentioned, this study necessitates a transformation of the fundamental equations and presents remarkable singularities.

The subject of this Note, therefore, is divided into two parts, the first of which, in view of its close relationship with Note II, is discussed in the same order and in the same manner.

PART 1ST

OPENING OPERATIONS IN REGIMEN.

§. 15. — Circular Diagrams of the Interlocked Series.

(Fig. 25 to 30)

Applying the graphic method of Fig. 3 (see § 6) to the determination of the ζ_i of the total rhytm in opening, it can be easily seen, by considerations analogous to those of § 8, and from the fact that the series ρ, $\rho\eta_1$, $\rho\eta_2$... is linearly increasing, that the resulting graphs have the following properties (Fig. 25 to 29).

1. — The centers C_1^*, C_2^*, C_3^*..., of the consecutive circles γ_1^*, γ_2^*, γ_3^*... are situated on a straight line at 45° and passing below the origin 0.

2. — The circles γ_1^*, γ_2^*, γ_3^*... all intersect at the same point M* of the bissectrix of the axes; the two coordinates of this point, equal ζ_m, represent the limiting value of the interlocked series ζ_i for an opening operation.

With the help of these properties, it is possible to study graphically the laws of the pressure in opening as follows:

Considering a circular diagram of the type of Fig. 25, it is easy to conclude that, if ρ is relatively large compared to θ (and in all cases when $\rho > 1$), all terms of the intelocked series of total rythme are larger than their limiting value ζ_m, and we have

$$1 > \zeta_1 > \zeta_2 > \zeta_3 > \ldots\ldots > \zeta_m ;$$

this results from the fact that the center C_1^* falls to the right of the vertical which is midway between the segments ζ_1 and ζ_m. The diagram of the pressure which tends assymptotically toward the limit ζ_m. therefore, has a form similar to Fig. 25bis. On the other hand, if the center C_1^* falls exactly on the vertical midway between ζ_1 and ζ_m, that is, if

$$\rho = \frac{1}{2}(1 + \zeta_m),$$

as shown on fig. 26 (where $\theta = 2$ and $\rho = 0.90$) we, of course, have

$$\zeta_1 = \zeta_2 = \zeta_3 \ \ldots = \zeta_m ,$$

which result is identical with the one obtained in § 8 for a closing operation the graph of the pressure therefore has the form of Fig. 26bis.

If ρ, with reference to θ, has a value, such that C_1^* falls to the left of the vertical midway between ζ_1 and ζ_m, and that the center C_2^* lies above M*, as shown in the case illustrated in Fig. 27, all terms of the interlocked series of total rythme are $< \zeta_m$, which can easily be ascertained. In this case the graph of the pressure must necessarily have the form of Fig. 27bis, i. e., during each phase, the pressure curve must twice intersect the horizontal of coordinate ζ_m^2. In fact, as this curve cuts the horizontal in question in B_1, it also must cut it in B_2, B_3, $B\zeta$..., etc., at intervals of one phase; but, because the pressures of the total rythme are always $< \zeta_m^2$, the curve also must intersect the horizontal coordinate of ζ_m^2 in a second series of points A_1, A_2, A_3.. etc., equally separated by intervals of one phase.

The law of pressure represented by fig. 27bis, therefore, has a form which is not related to any laws of the closure studied in Note II.

If now $\rho\eta_1$, i. e, the coordinate of point C_2^*, has a value between ζ_1 and ζ_m, that is, if we have approximately

$$\rho\eta_1 = \frac{1}{2}(\zeta_1 + \zeta_m),$$

(see fig. 28), it is clear that there results

$$\zeta_2 = \zeta_m .$$

and, consequently

$$\zeta_2 = \zeta_3 = \zeta_4 \ldots\ldots = \zeta_m ;$$

so that the graph of the pressure has the form of fig. 28bis.

Finally, if ρ is so small (also relative to θ) that a certain number of centers of odd indices C^*_1, C_3^*... are located to the left of the vertical through M^*, and a certain number of centers of even indices C_2^*, C_4^*... are below the horizontal through M^*, as illustrated in fig. 29, the intertocked values $\zeta_1, \zeta_2, \zeta_3$..., for the corresponding number of rythmes, are alternately $\lesseqgtr \zeta_m$, as can be easily proved from the positions of the circles $\gamma_1^*, \gamma_2^*, \gamma_3^*$... in fig. 29.

The graph of the pressure representing this condition has the characteristic form of fig. 29bis, which is analogous to fig. 4ter of closure (§ 8).

This form, however, does not continue beyond the rythme of uneven index for which C_i^* falls to the right of the vertical through M^*, or the rythme of even index, for which C_i^* falls bellow the horizontal trrough M^*; from there on we have curves of the form of fig's. 27bis and 28bis, in wich the pressures of the total rythme are constantly $\gtreqless 3_m^2$.

We shall see, in § 17, that the form of the pressure indicated in fig. 29bis cannot continue indefinitely during the whole time of the opening operation, except in the case where $\rho = 0$, that is, in the case of opening for the placing into service a conduit where the water is at rest.

The study of the relations which must exist between ρ and θ (and which determine the position of the conduit in the field of the synopsis) in order that the several cases represented by the fig. 's 25 to 29 could actually occur, can be accomplished in a thorough manner only by the analytical method; this study forms the subject of the following paragraph.

Before starting on this subject. I wish to call attention to an elementary, though extremely interesting, proprietary of the circular diagrams of interlocked series of closing and opening operations executed with equal speed (equal values of θ).

It is easy to see, that on this assumption, and the line c of the loci of the centers C_i for a closing operation, and the line c^*, locus of the centers C_i^* of an opening operation, are symmetrical with reference to the origin O (Fig. 30). If, therefore, as in fig. 30, we draw the circles γ_1 and γ_1^*, one corresponding to a closure from $\eta = 1$ to $\eta = \eta_1$, and the other to an opening operation from $\eta = \eta_1$ to $\eta = 1$, it can be seen that the points M and N of the bissectrix, which are symmetrical with reference to C and through which pass all the circles γ and the points M^* and N^*, which are symmetrical with reference to c^* and through which pass all the circles γ^* are correspondingly at equal distances from c and c^*.

The two points M* and N, therefore, are symmetrical with reference to the origin O, and the coordinates of N are, consequently, equal to the value of ζ_m of opening.

Moreover, plotting $\overline{OB_1} = 1$, and drawing the horizzontal $\overline{B_1E_1}$ to intersect with γ_1 in E_1, it is evident, that $\overline{B_1E_1} = \overline{A_1D_1} = \zeta_1$ for the opening of the amount from $\eta = \eta_1$ to $\eta = 1$. From the preceding it can be concluded, that the drawing of the single circle γ_1 give the following 4 values:

$\zeta_1^2 =$ pressure of the direct blow in closure, from $\eta = 1$ to $\eta = \eta_1 = \frac{\theta - 1}{\theta}$;

$\zeta_m^2 =$ limiting value of the interlocked series of pressures in closure;
$\zeta_1^2 =$ pressure of the direct blow in opening, from $\eta = \eta_1$ to $\eta = 1$,
$\zeta_m^2 =$ limiting value of the interlocked series of pressures in opening.
These 4 values are, of course, all referred to the same value of θ.

§ 16. — General laws of the pressure for opening from a condition of regime; corresponding synopsis.

(Fig. 31 to 33)

The analytical study of the general laws of the pressure of the total rythme in opening is derived from the study given in § 10, for the pressures of the total rythme in closure; the only difference being that $\pm\theta$ becomes $\mp\theta$. We have for closing

$$\eta_i = 1 - \frac{t}{\theta},$$

and for opening

$$\eta_i = 1 + \frac{t}{\theta};$$

It should be noted, however, that, while the duration of the closing gate operation can not exceed θ, the opening operation is by no means limited to this value.

It seems, therefore, that the analysis of the laws of pressure resulting from these two kinds of operation could have been treated in one single study; from certain points of view, for instance as regards the cartesian synopsis, this procedure would have the advantage of giving a more synthetic, more clear statement of the ensemble of laws which prevail during the phenomenon of the varying pressure.

However, on account of the novelty of the method, the complexity of the material, and also because of the technical importance of differentiating between the two subjects, I have first discussed the closing operations alone. Our studies will now be completed by extending them to the opening operations. It will be seen, by this very study, in what manner these two operations could have been discussed at the same time.

In a manner analogous to that demonstrated in § 10, it can be shown that, the series $1, \eta_1, \eta_2, \ldots$ being a series linearly increasing on the hypothesis of an opening operation, the interlocked series (both of the total and intermediate rythme) $\zeta_1, \zeta_2, \zeta_3 ....$ tend toward a limit ζ_m, determined by the equation obtained in substituting

$$\zeta_{i-1} = \zeta_i = \zeta_m \quad \eta_{i-1} = 1 + \frac{i-1}{\theta} \quad \eta_i = 1 + \frac{i}{\theta},$$

into the general equation (9)

$$\zeta^2_{i-1} + \zeta_i^2 - 2 = 2\rho\,(\eta_{i-1}\zeta_{i-1} - \eta_i \zeta_i).$$

In this manner we find

$$\zeta_m^2 + \frac{\rho}{\theta}\zeta_m - 1 = o, \tag{49}$$

which differs from (19) by the changed sign of the second member.

The value of ξ_m in opening, therefore, is < 1; moreover, it is a function of the single relation

$$\frac{\rho}{\theta} = \frac{Lv_0}{g\,\tau\,y_0},$$

and is independent of the value of a.

The equations (19) and (49) can, therefore, be considered as being forms of the same equation, on the condition that we attribute to θ the sign $\pm$, depending on dealing with an opening or closing operation.

The equation system derived from (9) & (49):

$$\frac{2\rho - (1 + \zeta_m)}{2\rho\eta_1 + \zeta_1 + \zeta_m} = \frac{\zeta_m - \zeta_1}{\zeta_m - 1} \tag{22}$$

$$\frac{2\rho\eta_1 - (\zeta_1 + \zeta_m)}{2\cdot\rho\eta_2 + \zeta_2 + \zeta_m} = \frac{\zeta_m - \zeta_2}{\zeta_m - \zeta_1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{2\rho\eta_{i-1} - (\zeta_{i-1} + \zeta_m)}{2\rho\eta_i + \zeta_i + \zeta_m} = \frac{\zeta_m - \zeta_i}{\zeta_m - \zeta_{i-1}}$$

is, therefore, applicable to an opening operation also; this system permits of determining the laws of the interlocked series of the pressures of total rythme for opening, and, consequently, their form with respect to the limiting value ξ_m^2.

CARTESIAN SYNOPSIS OF CLASSIFICATION.

As in the case of the closing operation, these laws, if plotted in the field of a cartesian synopsis, can serve to the classification of conduits, determined by lines which are represented by the equations obtained in equaling to 0 the numerators of equation (22); these equations, the general form of which

$$\zeta_i = \zeta_m, \quad \text{or} \quad 2\rho\eta_{i-1} = \zeta_{i-1} - \zeta_m \tag{24)*}$$

is identical with that of equation (24) of § 10, are but the equations of the curves s_i of § 11, in which the sign of the variable θ was changed.

Let us now assume that the cartesian synopsis, Fig. 31, is completed by the addition of a quadrant, the ordinates θ of which have an apposite sign to that assigned in the quadrant representing the closing operations. It is evident, that in the synopsis formed by the ensemble of these two quadrants, which we will designate the quadrant of closure and the quadrant of opening respectively, equation (24) represents the group of the branches of s_i located in the quadrant of closure, while equation (24)* represents the group of the branches of s_i situated in the quadrant of opening; these new branches we will, designate by s_i^*.

These loci s_i^* are situated between $\rho = \theta$ and $\rho = 1$, because in equation (24)* ρ must naturally be < 1; this results from the fact that, for an opening operation

$$\eta_i > 1, \zeta_i > 1, \zeta_m < 1 .$$

These curves, moreover, have $\rho = 1$ as their common asymptote, as it is evident that for $\theta = \infty$, $\text{Lim}\, \eta_i = 1$, $\text{Lim}\, \zeta_i = \text{Lim}\, \zeta_m = 1$, so that equation (24)* gives $\text{Lim}\, \rho = 1$.

Locus s_1^*.

The equation of this locus is:

$$\rho = \frac{1}{2}(1 + \zeta_m)$$

in which we must substitute the value of ζ_m from (49), which gives

$$\rho = \frac{4\theta + 1}{4\theta + 2}$$

This equation, as it should have been expected, is nothing else but equation (27) with a changed sign for θ; it represents the upper branch (Fig. 31) of the equilateral hyperbola, the lower branch of which is s_1 in the quadrant of closure.

The curve s_1^* touches the ρ axis at the point $\rho = 0.5$ and determines the left limit of the zone Σ_1^*, unlimited to the right, in which the trend of the law of pressure in opening is uniform.

All the curves $s_2^*, s_3^* . . s_i^*$, to the contrary, pass through the origin, as it is clear from equation (24)*, that if θ tends toward zero, η_{i-1} tends toward infinity and, consequently, ρ tends toward zero.

Locus s_2^*.

It has the equation

$$\rho\,\eta_1 = \frac{1}{2}(\zeta_1 + \zeta_m)$$

which can be put in the form

$$\rho\,(6\theta + 7) = 2\sqrt{\rho^2\,(\theta + 1)^2 + (2\rho + 1)\,\theta^2} + \sqrt{\rho^2 + 4\,\theta^2}$$

as per equation (29).

The locus passes through the points

$\theta =$	0,5	1,0	2,0	3,0	4,0	5,0	8,0	10	∞
$\rho =$	0,246	0,396	0,569	0,665	0,724	0,767	0,840	0,870	1,00.

The general equation of these loci is very complicated, but it can be replaced, with great approximation, by the limiting form

$$\rho\, \eta_{i-1} = \zeta_m, \text{ or}$$

$$\rho = \frac{\theta}{\sqrt{(\theta + i)^2 - (\theta + i)}} \tag{50}$$

which is equation (32) with the sign of θ changed.

From this equation we obtain the table

		s_3^*	s_4^*	s_5^*	s_6^*	s_8^*	s_{10}^*
$\theta = 0{,}5$	$\rho =$	0,169	0,126	0,105	0,084	0,063	0.050
» = 1	» =	0,289	0,224	0.183	0,154	0,118	0,095
» = 2	» =	0,447	0,365	0,309	0,267	0,211	0,174
» = 3	» =	0,548	0,463	0,401	0,353	0,236	0,240
» = 4	» =	0,617	0,534	0,471	0,422	0,348	0,296
» = 5	» =	0,668	0,589	0,527	0,476	0,401	0,345
» = 6	» =	0,707	0,632	0.572	0,522	0,445	0,387
» = 7	» =	0,738	0,667	0,609	0,561	0,483	0,424
» = 8	» =	0,763	0 696	0,641	0,593	0,516	0,458
» = 10	» =	0,801	0,701	0,690	0,645	0,572	0,513
» = 20	» =	0,889	0,851	0,816	0,745	0,737	0,678
» = ∞	» =	1.000	1.000	1.000	1.000	1.000	1.000

These values permit the construction of the synopsis of classification for the pressures of the total rythme in opening, Fig. 31, in which a portion of the synopsis in closure is also shown.

In order to conserve the analogy with the notation adopted for the sinopsis of closure, we will choose the symbols (see fig. 31):

Σ_1^* for the unlimited zone to the right of s_1^*

Σ_2^* » » zone between s_1^* and s_2^*

Σ_3^* » » » » s_2^* » s_3^*

in which zones the laws of the pressures in opening present different specific properties.

GENERAL LAWS OF THE PRESSURE IN OPENING.

We will study these laws with the help of formulas (22) and will refer to fig. 32, where the diagrams of the laws of pressure in opening are plotted for a series of conduits situated in the several zones Σ_i^* and also upon the lines of separation s_i^* of such zones, for the case $\theta = 2$.

Zone Σ_1^*.

For the conduits located in this zone, that is for those which satisfy the condition

$$\rho > \frac{1}{2}(\zeta_m + 1),$$

it is clear that the first of the equations (22) will give

$$(\zeta_m - \zeta_1) : (\zeta_m - 1) > 0$$

and, because $\zeta_m < 1$, we also have $\zeta_m < \zeta_1$.

The pressure of the direct blow, therefore, for all of this zone, is greater than the limiting pressure; it can be stated that this is equally true for the succeeding pressures of total rythme, so that the form of the pressure in opening will be simply asymptotical to the limiting pressure.

In fact, as already pointed out, if the two members of the first equation of (22) are $> o$, the two members of the succeeding equations will be also, a fortiori, (*) $> o$, which from the series of the second members, results in

$$\zeta_1 > \zeta_2 > \zeta_3 > \cdots\cdots > \zeta_m .$$

which proves the preceding statement.

Fig. 32 represents the form of the pressure in opening for the conduits characterized by

$$\theta = 2; \quad \rho = 1{,}90; \quad \rho = 1{,}5; \quad \rho = 1{,}15;$$

the line of these curves was determined very accurately by calculating the values of the intermediate rythme. The case $\rho = 1.5$ is the one illustrated in the circular diagram fig. 25.

In the same fig. 32 can be found also the diagram of the pressure in opening for the conduit $\rho = 0.9$, $\theta = 2$, situated on s_1^*, for which the conclusions precedingly stated hold good, namely

$$\zeta_1 = \zeta_2 = \zeta_3 = \cdots\cdots \zeta_m ;$$

the pressure diagram is made up of a series of arcs, concave downward, with the exception of the first one (see also fig. 26).

Zone Σ_2^*.

The conduits located in this zone satisfy the conditions

$$\frac{1 + \zeta_m}{2} > \rho > \frac{\zeta_1 + \zeta_m}{2\eta_1} ;$$

it follows that the two members of the first equation (22) are $< o$, while those of the second and all succeeding ones are $> o$. From the series of the second members it can be easily seen that

$$\zeta_1 < \zeta_2 < \zeta_3 < \cdots\cdots < \zeta_m ,$$

so that, in this case, the series of pressures of the total rythme is simply asymptotic to the limiting pressure, but this time, by smaller values.

Further, as already remarked in discussing fig. 27 where this case is illustrated, because the pressure has attained the value ζ_m^2 during the first phase, it must cross this same value twice during each succeeding phase; this explains the plot of the curve which consists of a series of arcs, concave downward, and which arcs cut the horizontal ordinate ζ_m^2; this result moreover can be easily verified analytically.

(*) Because $2\rho\eta_1 > 2\rho$ and $\zeta_1 + \zeta_m < 1 + \zeta_m$ the numerator of the first member of the second equation (22) must necessarily be $> o$ if the numerator of the first member of the first equation (22) is $> o$, and so fort.

Above this curve, in fig. 32, is shown that corresponding to the conduit $\theta = 2$ and $\rho = 0.569$, situated on s_2^* and for which

$$\zeta_1 < \zeta_m. \qquad \zeta_2 = \zeta_3 = \zeta_4 = \cdots\cdots = \zeta_m .$$

This case was discussed in conjunction with fig. 28.

Zone Σ_3^*.

The conduits in this zone satisfy the condition

$$\frac{\zeta_1 + \zeta_m}{2\eta_1} > \rho > \frac{\zeta_2 + \zeta_m}{2\eta_2};$$

from which it results that the two members of the first and second equation (22) are $< o$, while those of the third and following are $> o$.

From the series of the second members, it follows that

$$\zeta_1 < \zeta_m, \qquad \zeta_2 > \zeta_3 > \zeta_4 > \cdots\cdots > \zeta_m,$$

which signifies (see Fig. 32) that the pressure ζ_1^2 of the direct blow is smaller than the limiting value ζ_m^2, while the following pressures of the total rythme are greater than ζ_m^2; but, because the pressure must pass, during each phase through the value ζ_m^2 the third, fourth,... phases will have necessarily the form of arcs concave upward.

By extending the preceding results, we can evidently state for.

Zone Σ_i^*:

If the general condition

$$\frac{\zeta_{i-2} + \zeta_m}{2\,\eta_{i-2}} > \rho > \frac{\zeta_{i-1} + \zeta_m}{2\,\eta_{i-1}}$$

is satisfied, the conduit will be represented in the synopsis by a point located bweeen s_{i-1}^* and s_i^*. In this case:

1st: The pressures of the total rythme are alternately $\gtreqless \zeta_m^2$ upo to $t = i - 1$, that is

$$\zeta_1 < \zeta_3 < \zeta_5 < \cdots\cdots < \zeta_m$$
$$\zeta_2 > \zeta_4 > \zeta_6 > \cdots\cdots > \zeta_m .$$

2nd: From the instant $t = i - 1$

(a) if i is odd, that is if $\zeta_{i-1} > \zeta_m$ we will have

$$\zeta_{i-1} > \zeta_i > \zeta_{i+1} > \cdots\cdots > \zeta_m ;$$

b) if i is even, that is if $\zeta_{i-1} < \zeta_m$ we will have

$$\zeta_{i-1} < \zeta_i < \zeta_{i+1} < \cdots\cdots < \zeta_m .$$

These conclusions embrace, in their most general form, the laws of the pressure in opening; there seems to be no necessity to develop them further.

The only point which is really interesting from a technical point of view is the determination of the maximum depression resulting from a given opening operation; now, it follows clearly from the preceding study that:

A. For all conduits located in the zone Σ_1^*, to the right of s_1^*, the minimum pressure is the one which occurs at the end of the gate operation, and this pressure differs but little from ζ_m^2 if the operation has a sufficiently long duration.

B. For all conduits located in the zones $\Sigma^*_{2,3,4}$, etc. to the left of s_1^*, the minimum pressure is the pressure of the direct blow, assuming that the operation has a length of duration of at least one phase.

It is, moreover, entirely impossible that the minimum pressure would occur at instants of the intermediate rythme. For this reason we will dispense with the study of this subject; the only synoptic diagram which presents real technical interest is, therefore, that of the pressures of the direct blow ζ_1^2 and the limiting pressures ζ_m^2.

SYNOPTIC DIAGRAM OF THE PRESSURES ζ_m^2 AND ζ_1^2.

The construction of this diagram (analogous to the fig. 19 and 20 for clo sure) follows naturally from the plots fig. 33; it embraces:

A). The series of equilateral hyperbolas passing through the origin; this system of curves is characterized by the first equation of (22)

$$\zeta_1^2 - 1 = 2\,\rho\,(1 - \eta_1\,\zeta_1),$$

from which, giving to ζ_1^2 a series of constant values < 1, and putting

$$\eta_1 = (\theta + 1) : \theta$$

we obtain

$$\rho = \frac{1 - \zeta_1^2}{2}\,\frac{\theta}{\zeta_1 - (1 - \zeta_1)\,\theta}; \qquad (51)$$

this is the equation of the system of curves, analogous to the equation (35) of § 12 relating to a closing operation.

B). The system of straight lines issuing from the origin, characterized by the equation (49)

$$\zeta^2_m + \frac{\rho}{\theta}\,\zeta_m - 1 = 0\,,$$

in which constant values < 1 are assigned to ζ_m^2 and from which follows

$$\frac{\rho}{\theta} = \zeta_m^{-1} - \zeta_m\,; \qquad (52)$$

This is the general equation of the series of lines analogous to equation (34) of § 12.

The system of the hyperbolas (51) gives, therefore (Fig. 33) the loci of conduits, for which the pressure of the direct blow, in the zones $\Sigma_{2,3,4 \ldots i}$ to the left of s_1^*, has a given value which is the minimum pressure occurring during the gate operation; the system of radiating straight lines in the space Σ_1^*, to the right of s_1^*, gives, to the contrary, the loci of conduits for which the minimum pressure in opening has a given value, which value is smaller than the minimum which can occur during the gate operation.

Conforming to the opening remarks of this paragraph, in the synopsis Fig. 33 (as in fig. 31) an opposite sign was given to θ than that of the θ of closure; in this manner a cartesian rèpresentation was obtained in which the opening operations occupy a quadrant separate from that indicating closing operations. A portion of this latter is shown in the fig's. 31 and 33.

The form of these two synopsis demonstrates, as already remarked, that the problems relating to waterhammer in closure and in opening could be condensed in the same single study.

PART 2[nd]

OPENING FOR PLACING IN SERVICE.

§ 17. Formulas and general laws of the pressure in opening for placing in service.

In the case of a conduit originally closed we have

$$v_0 = o, \qquad \psi_0 = \frac{v_0}{u_0} = o,$$

from which would result:

$$\rho = \frac{av_0}{u_0^2} = o, \qquad \eta_i = \frac{\psi_i}{\psi^0} = \infty,$$

while, in the relation

$$\eta_i = 1 + \frac{t}{\theta},$$

we would have

$$\theta = \frac{t}{\eta_i - 1} = 0.$$

If, therefore, we assume v_0 to be very small and approaching zero, θ and ρ are very small also and approach zero; it appears, then, that all formulas and graphs derived in the preceding paragraphs must lose all significations.

However, this is not the case, because if, in this assumption, the limits $\rho = 0$ and $\theta = 0$ are reached, the products which figure in the fundamental equation system (9), are, nevertheless, finite quantities: in fact, we have

$$\rho\,\eta_i = \frac{av_0}{u_0^2} \cdot \frac{\psi_i}{\psi_0} = \frac{a\,\psi_i}{u_0}, \tag{53}$$

which is a finite quantity. This observation permits the modification of the whole system of our formulas by introducing in then, instead of the symbols defining the initial condition, the symbols which define the final regimen toward which the conduit tends.

For this purpose, let

$v_* =$ the velocity of regimen, corresponding to the degree of opening attained at the end of the operation;

$\theta_* =$ the duration of the opening operation;

$\rho_* = \dfrac{a\,v_*}{u_0^2}$, the characteristic corresponding to the regimen;

$\psi_* = \dfrac{v_*}{u_0}$, the ratio of the gate opening to the section of the conduit;

$t_* =$ any instant in the first phase $(t_* < \mu)$.

It is evident, that at the instants

$$t_* \qquad t_* + \mu \qquad \ldots\ldots \qquad t_* + i\,\mu,$$

we will have

$$\psi_1 = \frac{t_*}{\theta_*}\,\psi_* \qquad \psi_2 = \frac{t_* + 1}{\theta_*}\,\psi_* \qquad \ldots\ldots \qquad \psi_i = \frac{t_* + i - 1}{\theta_*}\,\psi_*$$

from which, by means of equation (53)

$$\rho\,\eta_i = \frac{a\,\psi_i}{u_0} = \frac{a v_*}{u_0^2} \cdot \frac{t_* + i - 1}{\theta_*} = (t_* + i - 1)\,\frac{\rho_*}{\theta_*} \tag{54}$$

Making successively $i = 1, 2, 3, \ldots$ etc., and substituting (54) into the fundamental system (9), we obtain:

$$\begin{aligned} \zeta_1^2 - 1 &= -2\,\frac{\rho_*}{\theta_*}\,t_*\,\zeta_1 \\ \zeta_1^2 + \zeta_2^2 - 2 &= 2\,\frac{\rho_*}{\theta_*}\Big(t_*\,\zeta_1 - (t_* + 1)\,\zeta_2\Big) \\ \zeta_2^2 + \zeta_3^2 - 2 &= 2\,\frac{\rho_*}{\theta_*}\Big((t_* + 1)\,\zeta_2 - (t_* + 2)\,\zeta_3\Big)\,. \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{55}$$

This is the specific form of the fundamental system for the interlocking series of any intermediate rhythm, in the case of an opening operation starting from a complete closure.

If we put $t_* = 1$, we obtain the interlocked series of the total rhythm; the system (55) then takes the simpler form

$$\begin{aligned} \zeta_1^2 - 1 &= -2\,\frac{\rho_*}{\theta_*}\,\zeta_1 \\ \zeta_1^2 + \zeta_2^2 - 2 &= 2\,\frac{\rho_*}{\theta_*}\,(\zeta_1 - 2\,\zeta_2) \\ \zeta_2^2 + \zeta_3^2 - 2 &= 2\,\frac{\rho_*}{\theta_*}\,(2\,\zeta_2 - 3\,\zeta_3)\,. \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{56}$$

By means of the systems (55) and (56) we now can make a full study of the laws of pressure for the placing of the conduit in service.

Let, in the general equation (55)

$$\zeta_{i-1} = \zeta_i = \zeta_m,$$

and we get

$$\zeta_m^2 + \frac{\rho_*}{\theta_*}\,\zeta_m - 1 = 0 \tag{57}$$

which equation gives the limiting value ζ_m of the interlocked series, and which, in substance, is identical with equation (49) of § 16.

The cause of this identity is evident, because if the opening operation would continue indefinitely (that is beyond the degree of opening to which the characteristic corresponds) the equation which would furnish the limiting value of the pressure would be precisely equation (57).

GENERAL LAWS OF THE PRESSURE.

In the case of an opening operation for placing a conduit in service, the general laws of the pressure are the same as those for the case of opening of conduits in service located to the left of s_1^* (Fig. 31 and 32), for which the maximum depression is the one which occurs at the end of the first phase (depression of the direct blow).

It suffices to compare the first equation of (56) with equation (57) to be convinced, in fact, that in the case of an opening for placing in service $\zeta_1 < \zeta_m$ always.

Another particularity of this kind of opening is that the pressure takes the value ζ_m^2 exactly in the middle of the first phase, and, consequently, takes the same value in the middle of the consecutive phases: In fact, putting in the first equation of (55), $t = 0.50$, we obtain, for the instant of the intermediate rhythm of the middle of the first phase:

$$\zeta_1^2 + \frac{\rho_*}{\theta_*}\zeta_1 - 1 = 0 \qquad (58)$$

which equation, compared with (57) shows precisely that in the middle of the first phase $\zeta_1 = \zeta_m$ and that, consequently, we will have, in the middle of the other phases $\zeta_2 = \zeta_3 = \zeta_3 ... = \zeta_m$.

We can continue the study of the pressure law in a manner analogous to those of the preceding paragraphs, by seeking the law of the interlocked series of the pressures of the total rythm.

Combining equation (57) whith the equations (56), except the first, and putting, for sake of simplicity

$$\varepsilon_* = \rho_* : \theta_*$$

we obtain

$$\begin{aligned} \frac{2\varepsilon_* - (\zeta_1 + \zeta_m)}{4\varepsilon_* + \zeta_2 + \zeta_m} &= \frac{\zeta_m - \zeta_2}{\zeta_m - \zeta_1} \\ \frac{4\varepsilon_* - (\zeta_2 + \zeta_m)}{6\varepsilon_* + \zeta_3 + \zeta_m} &= \frac{\zeta_m - \zeta_3}{\zeta_m - \zeta_2} \qquad (59) \\ \frac{6\varepsilon_* - (\zeta_3 + \zeta_m)}{8\varepsilon_* + \zeta_4 + \zeta_m} &= \frac{\zeta_m - \zeta_4}{\zeta_m - \zeta_3} \end{aligned}$$

etc.

But we know that in the case of an opening for placing in service we always have

$$\zeta_m - \zeta_1 > 0;$$

the signs of the numerators of the first members of equation (59) will determine the signs of the successive values of $\zeta_m - \zeta_2$, $\zeta_m - \zeta_3$... etc.

First, if

$$2\varepsilon_* - (\zeta_1 + \zeta_m) > 0,$$

all the numerators of the first members of equations (59) will, a fortiori, be > 0; it is, then, easy to conclude, that

$$\zeta_1 < \zeta_2 < \zeta_3 < \cdots\cdots < \zeta_m,$$

so that the diagram of the pressure will have the form of that in fig. 32, for a conduit situated in zone Σ^*_2, where the arcs are concave downward and where the pressures of the total rythm are $< \zeta_m{}^2$ (see also fig. 34[bis]).

If, on the contrary:

$$2\,\varepsilon_* - (\zeta_1 + \zeta_m) < 0,$$

while the numerators of the succeeding equations are > 0, it is clear, that

$$\zeta_2 > \zeta_2 > \zeta_4 \cdots\cdots > \zeta_m,$$

the diagram then will have the form of that in fig. 32, for a conduit situated in zone $\Sigma_3{}^*$, where the arcs are concave upward and where the pressures of the total rythm are $> \zeta_m{}^2$, beginning at $\zeta_2{}^2$.

Finally, if

$$2\,\varepsilon_* - (\zeta_1 + \zeta_m) = 0,$$

we have evidently

$$\zeta_2 = \zeta_3 = \zeta_4 = \cdots\cdots = \zeta_m\,;$$

the pressure takes the value $\zeta_m{}^2$ in the middle of each phase and also at the instants joining the phases. The line of pressure, therefore, in the interval of one phase, will have a point of inflection (see fig. 35[bis]).

From the preceding, without extending these considerations to more phases, the following conclusion can be made:

If a certain number of numerators of the first members of equation (59) are < 0, the pressures of the total rythm are alternately $\lesseqgtr \zeta_m$ for a corresponding number of phases; the succeeding pressures of the total rythm, however, are all $> \zeta_m{}^2$ or all $< \zeta_m{}^2$, depending on whether the number of equations (59) the two members of which are $< o$, is odd or even.

Finally, if any of the numerators in question is zero, that is, if

$$2\,(i-1)\,\varepsilon_* = \zeta_{i-1} + \zeta_m \qquad (60)$$

we will have

$$\zeta_i = \zeta_{i+1} = \zeta_{i+2} = \cdots\cdots = \zeta_m.$$

We will see below (see the observation at the end of this paragraph), how te condition expressed by (60) can be synoptically represented in terms of ρ^* and θ^*.

CIRCULAR DIAGRAM OF THE INTERLOCKED SERIES $\zeta_1, \zeta_2 .., \zeta_i$.

With the notation $\varepsilon_* = \rho_* : \theta_*$

the equation system (56) can be written

$$\begin{aligned}
\zeta_0{}^2 + (\zeta_1 + \varepsilon_*)^2 &= \varepsilon_*{}^2 + 2\\
(\zeta_1 - \varepsilon_*)^2 + (\zeta_2 + 2\,\varepsilon_*)^2 &= 5\,\varepsilon_*{}^2 + 2\\
(\zeta_2 - 2\,\varepsilon_*)^2 + (\zeta_3 + 3\,\varepsilon_*)^2 &= 13\,\varepsilon_*{}^2 + 2\\
\cdots\cdots\cdots\cdots
\end{aligned} \qquad (61)$$

from which it appears that equations, (61) in the coordinates ζ_{i-1} and ζ_i represent a system of circles $\gamma_1{}^*, \gamma_2{}^*$... etc. having for centers an radii respectively

		Centres		(Radii)²
$\gamma_1{}^*$	$C_1{}^*$:	$(0),$	$(-\,\varepsilon_*)$	$\varepsilon_*{}^2 + 2$
$\gamma_2{}^*$	$C_2{}^*$:	$(+\,\varepsilon_*),$	$(-\,2\,\varepsilon_*)$	$(1 + 2^2)\,\varepsilon_*{}^2 + 2$
$\gamma_3{}^*$	$C_3{}^*$:	$(+\,2\,\varepsilon_*),$	$(-\,3\,\varepsilon_*)$	$(2^2 + 4^2)\,\varepsilon_*{}^2 + 2$
...	...	...	...	...
$\gamma_i{}^*$	$C_i{}^*$:	$((i-1)\,\varepsilon_*),$	$(-\,i\,\varepsilon_*)$	$((i-1)^{2*} + i^2)\,\varepsilon^{*2} + 2$

It is evident that this system can be derived, without other consideration, from that mentioned in § 15, which is illustrated in fig's 25 to 30, in putting

$$\rho = o, \qquad \rho\, \eta_1 = \varepsilon_*, \qquad \rho\, \eta_2 = 2\, \varepsilon_*, \text{ etc.}$$

In the case of placing a conduit in service, therefore, the circular diagram of the interlocked series is characterized by the fact that the first center is located on the vertical axis, at a point where the ordinate is $-\varepsilon_*$ (see fig's. 34, 35, 36).

Fig's. 34 and 34bis illustrate the case, where, the numerator of the first member of the first equation (59) being a positive quantity, the diagram of the pressure, beginning at the first phase, has a form of a series of arcs concave downvard, and where all pressures of the total rhythm are $< \zeta_m^2$.

Fig's. 35 and 35bis represent the case in which the numerator mentioned being $= 0$, all pressures of the total rythm $= \zeta_m^2$. The condition

$$2\, \varepsilon_* - (\zeta_1 + \zeta_m) = o,$$

is evidently satisfied by a single value of ε_*, in other words by a single operation.

Introducing the values

$$\zeta_1 = \sqrt{\varepsilon_*^2 + 1} - \varepsilon_* \quad \zeta_m = \frac{1}{2}\left(\sqrt{\varepsilon_*^2 + 4} - \varepsilon_*\right)$$

derived from the first of equations (56) and (57), we easily arrive at the value

$$_* = 7 : \sqrt{120} = 0{,}64 \text{ environ.}$$

Finally, Fig's. 36 and 36bis illustrate the case where the numerator of the first member of the first equation (59) being negative, it happens that a certain number of the pressures of the total rhythm are alternately $\gtrless \zeta_m^2$.

SUDDEN OPENING

Considering the first equation of (55)

$$\zeta_1^2 + 2\frac{\rho_*}{\theta_*}\, t_*\, \zeta_1 - 1 = 0,$$

we will call « sudden opening » all such operation which brings the conduit to a degree of opening corresponding to the final regime in a time $\theta_* = t_*$, or in other words, an operation performed in a time $\theta_* < 1$, that is, during the phase of the direct blow.

On this assumption, the first equation of (55) becomes:

$$\zeta_1^2 + 2\, \rho_*\, \zeta_1 - 1 = 0, \qquad (62)$$

from which follows:

$$\zeta_1 = \sqrt{\rho_*^2 + 1} - \rho_* \qquad (62\ bis)$$

Applying this equation (62bis) to the sudden placing in service of conduits for the values of the characteristic of regime varying between 0.10 and 10, we obtain.

$\zeta_* =$ 0,10	0,25	0,50	1,00	2,00	3,00	5,00	10,00
$\zeta_1 =$ 0,819	0,781	0,618	0,414	0,236	0,162	0,099	0,005
$\zeta_1^2 =$ 0,670	0,610	0,380	0,170	0,056	0,026	0,010	0,000

The depression, therefore, is the greatest, the largest the characteristic, that is when the head is least, which result, of course, could be expected.

For heads of 10 to 15 met. ($\rho_* = 5$ to 10), the sudden opening reduces the pressure pratically to zero, but this phenomenon, as will be seen in Note IV, does not result in daagerous counter blows.

CARTESIAN SYNOPSIS OF ζ_e^2 (Fig. 37)

We have stated that the maximum depression due to an opening for placing in service is always given by ζ_1^2, the pressure of the direct blow; this statement limits the technical interest in the cartesian synopsis, in ρ_* and θ_*, to the representation of the loci of conduits for which ζ_1^2 has a given value.

These loci are, of course, given by the first equation of (56) and are characterized by

$$\frac{\rho_*}{\theta_*} = \frac{1}{2}(\zeta_1^{-1} - \zeta_1),$$

which represents a series of straight lines passing through the origin; we have, in this manner

$\zeta_1^2 =$	0,9	0,8	0,7	0,6	0,5	0,4	0,3	0,2	0,1	0,05	0,02
$\frac{\rho_*}{\theta_*} =$	0,052	0,112	0,179	0,257	0,353	0,475	0,638	0.895	1,424	2,120	3,475

The figures of this table were used to construct the synopsis fig. 37., which consists entirely of straight lines and which therefore in remarkably simple.

In the zone between $\theta_* = 0$ and $\theta_* = 1$, which is the zone of sudden opening, the loci of the conduits, for which ζ_1^2, due to an opening for placing in service, has a given value, are, of course, recti-linear vertical segments.

REMARK

The fact that, in the synopsis ρ_*, θ_* the loci $\zeta_1^2 = \text{const.}$, are straight lines and not hyperbolas, suggests an interesting observation. As pointed out in the beginning of this §, if v_0 is very small, and becomes, in the limit, equal to zero, ρ and θ also are very small and tend equally toward the limit zero. Consider now the cartesian synopsis in ρ and θ for an opening operation (Fig. 31 and 33); it is evident that the conduits for which $\text{Lim}\ \rho = 0$ and $\text{Lim}\ \theta = 0$ will be represented by the points of a zone situated in the corner of the coordinate axes, and which zone, in the limit, becomes also infinitely small.

The adoption of the new parameters ρ_* and θ_*, instead of ρ and θ confers a finite value upon the terms of the formulas, and, therefore, also gives finite dimensions to be used in the synoptic representation of conduits operated for placing them into service.

The plot of the lines (63) passing through the origin, which, as we have seen, are the loci of the conduits for which $\zeta_1^2 = \text{const.}$, reproduces, with finite dimensions, the infinitesimal arcs which constitute the linear elements of the equilateral hyperbolas (51), in the vicinity of the origin; this statement is proved by the fact that the straight lines in question are precisely the tangents of the hyperbolas at the origin, which can be easily seen by differentiatig equation (51).

$$\left(\frac{\delta\rho}{\delta\theta}\right)_{\theta=0} = \frac{1}{2}(\zeta_1^{-1} - \zeta_1).$$

Finally, in fig. 37. it is equally possible to draw a plot of straight lines issuing from the origin and reproducing, with finite dimensions, the elementary arcs of the loci s_1^* in the vicinity of the origin; these lines are the loci of those conduits for which the law of pressure due to an opening operation for placing in service is characterized by the vanishing of the numerator of one of the first members of equations (59), that is, by

$$2(i-1)\varepsilon_* = \zeta_{i-1} + \zeta_m, \tag{60}$$

which is the general equation of this system of straight lines.

This system which could be called the plot s_1^*, would divide the synoptic quadrant into angular spaces Σ_2^*, Σ_3^*..... comprising the conduits for which the laws of the pressure in opening have a form similar to the corresponding zones of the synopsis Fig. 31, which were sufficiently developed.

However, this synopsis of classification of the conduits from the point of view of opening operations for placing in service would have a purely theoretical interest only; it was, therefore, omitted from fig. 37.

NOTE IV.

COUNTER BLOW DURING RETURN TO REGIMEN

§ 18. — Formulas and general laws of the variable motion with gate at rest

If the gate stops after an operation of closure or opening has induced a disturbed regimen in the conduit, the hydrodynamic phenomenon produced from the time of stoppage must necessarily tend asymptotically toward the new conditions of permanent regimen depending on the degree of opening attained.

Let us designate by η_* and ζ_* the values of η and ζ relative to an istant t_* of the first phase counted from the moment that the gate has stopped moving; it is then evident that the interlocked series $\zeta_1, \zeta_2, \zeta_3 \ldots$, etc., corresponding to the instants $t_* + \mu$, $t_* + 2\mu$, $t_* + 3\mu \ldots$ will be defined by the system (9), in which we substitute $\eta_1 = \eta_2 = \eta_3 \ldots = \eta_*$, $\rho\eta_* = \rho_*$.

We have in this manner

$$\begin{aligned} \zeta^2_* + \zeta^2_1 - 2 &= 2\,\rho_*\,(\zeta_* - \zeta_1) \\ \zeta_1^2 + \zeta_2^2 - 2 &= 2\,\rho_*\,(\zeta_1 - \zeta_2) \\ \zeta_3^2 + \zeta_2^2 - 2 &= 2\,\rho_*\,(\zeta_2 - \zeta_3) \\ \ldots\ldots&\ldots\ldots \end{aligned} \qquad (61)$$

while ρ_* designates the characteristic of the new regimen.

It is also evident that lhe limiting value of this interlocked series must be $\zeta_m = 1$, which value satisfies thet general equation (61) in making

$$\zeta_{i-1} = \zeta_i = \zeta_m,$$

The circular diagram of the interlocked series at stopped gate (Fig. 38 to Fig. 41) illustrates in a simple and elegant form the laws of the return to regimen.

In this case, this diagram reduces to two circles γ_1 and γ_2 of centers C_1, (coordinates $+\rho_*$ and $-\rho_*$) and C_2 (coord. $-\rho_*$ and $+\rho_*$), of a radius $\sqrt{2\rho^2_* + 2}$ and located symmetrically with reference to the bissectrix of the axes. These figures clearly indicate that the interlocked series $\zeta_1, \zeta_2, \zeta_3 \ldots$ tend toward the value of the coordinate of the point K, that is toward the limit $\zeta_m = 1$, which realizes the new state of regimen.

But this new state of regimen can not be obtained if the gate operation is pushed to the complete closure of the orifice; in this case $\eta_* = 0$ and $\rho_* = 0$, and the system (61) becomes

$$\begin{aligned} \zeta_*^2 + \zeta_1^2 - 2 &= 0 \\ \zeta_1^2 + \zeta_2^2 - 2 &= 0 \\ \ldots\ldots&\ldots \end{aligned} \qquad (62)$$

and the pressure oscillates indefinitely between the limits ζ_*^2 and $2 - \zeta_*^2$.

The two circles γ_1 and γ_2, in this case, become one, the circle $\gamma\gamma$ of center O and radius $\sqrt{2}$ (fig. 46), and the point of this circle with coordinates ζ_* and $\sqrt{2-\zeta_*^2}$ characterize the limiting pressures.

Returning now to equation system (61), we find, in writing same in the form:

$$\begin{aligned} \frac{2\rho_* - (\zeta_* + 1)}{2\rho_* + (\zeta_1 + 1)} &= \frac{\zeta_1 - 1}{\zeta_* - 1} \\ \frac{2\rho_* - (\zeta_1 + 1)}{2\rho_* + (\zeta_2 + 1)} &= \frac{\zeta_2 - 1}{\zeta_1 - 1} \\ \dots\dots & \dots\dots \end{aligned} \qquad (63)$$

that it is the same as equation (22), of Note III.

The several cases of the return to regimen which may occur depend exclusively on the sign of the numerator of the first equation (63).

First case.

$$2\rho_* - (\zeta_* + 1) < 0 \text{ or } \rho_* < \frac{1}{2}(\zeta_* + 1).$$

It is evident that in this case the two members of the first equation of the system (63) are negative, so that if we have:

$$\zeta_* > 1 \qquad \text{it must be that} \qquad \zeta_1 < 1 \qquad \text{(fig. 38)}$$

and, inversely, if

$$\zeta_* < 1 \qquad \text{it must be that} \qquad \zeta_1 > 1 \qquad \text{(fig. 39)}$$

On both assumptions the pressure must cross the limiting value $\zeta^2_m = 1$, at a certain instant of the first phase following the stoppage, and must take the same value $\zeta_m^2 = 1$ at intervals μ counted from this instant.

In this first case, therefore, we can conclude that the pressure takes an oscillatory character tending asymptotically toward its value of regimen. Figures 38 and 39 illustrate these cases for the assumption $\zeta_* \gtreqless 1$.

Second case.

$$2\rho_* (\zeta_* + 1) = 0; \text{ or } \rho_* = \frac{1}{2}(\zeta_* + 1).$$

Evidently we have then

$$\zeta_1 = \zeta_2 = \zeta_3 \dots\dots = 1,$$

so that the pressure reaches its value of regimen at the first instant of total rhytm succeeding the stoppage of the gate, and retains this value, from this time on.

Third case.

$$2\rho_* - (\zeta_* + 1) > 0; \qquad \rho_* > \frac{1}{2}(\zeta_* + 1).$$

In this case the two members of the first equation (63) are positive, and it can be easily seen from the succeeding equations of the same system (63), that

if $\zeta_* > 1$, it must be that $\zeta_* > \zeta_1 > \zeta_2 > \cdots > 1$ (fig. 40)
if $\zeta_* < 1$, it must be that $\zeta_* < \zeta_1 < \zeta_2 < \cdots < 1$ (fig. 41).

On both of these assumptions, the pressure tends asymptotically and without oscillations toward its value of regimen (*).

The formulas and graphs discussed above solve, in a complete manner, and in a most general form, the problem of the laws of pressure of the counter blow of return to regimen; it is believed, nevertheless, that it is well to make a complete systematic study of these laws for the typical cases of counter-blow succeeding either a closing or an opening gate operation.

This study, which will be the subject of § 20 and § 21, must, however, be preceded by one of more general character, aiming to determine the conditions under which the analytical expression of the pressure can take zero, negative or imaginary values, and to define the significance to be attributed to such results.

This last research, in fact, could have been incorporated in Note I; it seems, however, better placed here because, in view of the very nature of the phenomenon, it is rather closely related to the study of the counterblow with stopped gate; it forms the subject of § 19 below.

§ 19. — General conditions for the pressure becoming zero, negative or imaginary during a gate operation.

The general equation characterizing the term ζ_i of an interlocked series being of the form:

$$\zeta_i^2 + 2\,\rho\eta_i\,\zeta_i - C_i = o,$$

rom which

$$\zeta_i = \sqrt{\overline{\rho\eta_i}^2 + C_i} - \eta_i$$

it is evident that, in order that ζ_i should have a positive, real value, C_i must be $> o$. We will now examine the cases for which, this condition not being fulfilled, ζ_i may have a zero, negative or imaginary value.

First case.

$$C_i = o, \qquad \zeta_i = o$$

In this case, the physical significance of the formulas is very clear; we have measured the pressures Y_i in meaters above the atmospheric pressure, therefore the pressure determined by $\zeta_i^2 = o$, $Y_i = o$ means atmospheric pressure. Should this pressure be measured from vacuum, its numerical value would be about 10 meters.

Second case:

$$C_i < o, \qquad \overline{\rho\eta_i}^2 > C_i, \qquad \zeta_i < o.$$

It should be remarked that ζ_i and ζ_i^2 must have the same sign; so that if it is found from the formulas that ζ_i is negative, then ζ_i^2 must be negative also. This apparent deviation from the algebraic conventions follows from the laws of the physical phenomenon of the flow of fluids.

(*) Therefore, there does not exist a fourth case, in which the pressure, after a first oscillation, becomes asymptotic to its value of regimen, to which erroneous conclusion I arrived in my monograph of 1904.

Let us assume, for sake of argument, that we have two reservoirs of different levels, separated by a thin wall in which there is an opening. The relation $u = \sqrt{2gy}$, giving the velocity of flow from the first reservoir to the second, should be written in the form $u = -\sqrt{2gy}$ and not $u = \sqrt{-2gy}$, if y is negative, i.e., if the second reservoir had higher waterlevel than that of the first.

In the present case, the significance of the formulas is perfectly clear and simple as long as the pressure $-\zeta_i^2 y_0$ does not exceed in absolute value the numerical value of 10 meters (atmospheric pressure) i.e., as long as the pressure measured from absolute vacuum is positive; per contra, if it exceeds this value, the formulas will have no physical significance at all, unless, by hypothesis, we would attribute to the fluid physical properties which would permit tensional stresses to exist both in the body of the fluid and between the fluid and the walls of the pipe.

Such fluids, however, do not exist in nature; therefore, the preceding formulas, when they result in sub-pressures the absolute value of which are greater than the atmospheric pressure, indicate that discontinuities are produced within the fluid mass. Perturbations must, therefore, result of such character, that the produced hydro-dynamic phenomena do not obey any more the laws of the interlocked series, expressed by the fundamental system.

Third case:

$$C_i < o, \qquad (\rho\eta_i)^2 < C_i, \qquad \zeta_i \text{ imaginairy.}$$

The fact that ζ_i and, consequently, ζ_i^2 are imaginary, can not mean, in the author's opinion, anything else but that, in this case, no hydro-dynamic equilibrium of any order is possible, no matter what proprieties are attributed hypothetically to the fluid. In practice, therefore, we revert to case 2, where $-\zeta_i^2 y_0 > 10$, meaning that, due to discontinuities the phenomenon is beyond the laws of the interlocked series.

The circular diagram of the interlocked series applied to these several cases will evidently furnish a negative segment representing ζ_i in the case of a negative pressure ζ_i^2; in the case of an imaginary pressure, per contra, this diagram does not give any value for ζ_i because, as shown in an example below, the circle γ_i is not intersected by the straight line on which should be found the segment representing ζ_i.

In order to better illustrate the preceding results, we are now going to find the general conditions which must be satisfied so that the first counter blow ζ_2^2 be zero, negative or imaginary, reserving for the succeeding paragraph a systematic establishment of the laws of the counterblow resulting from the diverse gate operations.

CONDITIONS THAT $\zeta_2^2 \leq 0$.

From the two first equations of (9)

$$\begin{aligned} \zeta_1^2 - 1 &= 2\rho(\ \ 1 - \eta_1\zeta_1) \\ \zeta_1^2 + \zeta_2^2 - 2 &= 2\rho(\eta_1\zeta_1 - \eta_2\zeta_2) \end{aligned} \qquad (9)$$

we get, by subtraction

$$\zeta_2^2 + 2\rho\eta_2\zeta_2 - 4\rho\eta_1\zeta_1 + 2\rho - 1 = 0;$$

The condition for $\zeta_2 \gtrless 0$ is, therefore;

$$-4\rho\eta_1\zeta_1 + 2\rho - 1 \gtrless 0,$$

and, eliminating ζ_1 by means of the first equation of (9)

$$\eta_1 \lessgtr \frac{2\rho - 1}{2\rho\sqrt{4\rho + 6}} \tag{64}$$

From this equation the following table results:

for $\rho = 0{,}5$	0,75	1,0	2,0	3,0	4,0	5,0	10,0	∞
$\eta_1 = 0$	0,111	0,158	0,200	0,196	0,186	0,155	0,140	0
$\theta = 1$	1,125	1,162	1,250	1,244	1,228	1,183	1,163	1

In order that the phenomenon characterized by $\zeta_2^2 \lessgtr 0$ should occur, it is therefore, necessary that the speed of closure, in the first phase be very rapid,

The minimum of this speed of closure (or the maximum of θ) occurs about for $\rho = 2$, or, more exactly for

$$\rho = \frac{1}{4}(\sqrt{33} + 3) = 2{,}1865,$$

which value is easily obtained by differentiating equation (64):

$$\frac{\delta\eta_1}{\delta\varsigma} = 2\rho^2 - 3\rho - 3 = 0.$$

Figures 42, 43, 44, drawn for $\rho = 2$, and for values

$$\eta_1 < 0{,}2, \qquad \eta_1 > 0{,}2, \qquad \eta_1 = 0{,}2,$$

illustrate the preceding conclusions by the help of the circular diagrams of the interlocked values of ζ_1 and ζ_2.

A remarkable, and on first thought, paradoxical statement follows from the fact that, when η_1 has the value given ny (64), the pressure of the counter blow ζ_2^2 is zero, no matter what is the value of η_2, that is, whatever may be the operation which follows that of the partial closure executed in the first phase.

Figures 42, 43 and 44 easily solve this paradox. In fact, $\rho\eta_1$ being given, it is easy to demonstrate (*) that for any value of η_2 that is, for any position of C_2 on the horizontal coordinate $\rho\eta_1$, the circle γ_2 always passes through the same point B_2 of the vertical axis; therefore, as in the case in fig. 44 we have $\zeta_1 = OB_2$, it is clear that, equation (64) being satisfied, we have $\zeta_2 = 0$ for any value of η_2.

If, to the contrary, ζ_2 is different from 0, it varies with η_2 in such a manner that it increases in absolute value as η_2 decreases, as shown in figs. 42 and 43; and can also become imaginary.

CONDITIONS THAT ζ_2^2 BECOMES IMAGINARY

The condition that the first pressure of the counter blow shall become imaginary is naturally:

$$-4\rho\eta_1\zeta_1 + 2\rho - 1 > \rho^2\eta_1^2$$

which condition implies that of the inequality (64). However, no conclusion can be derived frnm this condition without making assumption regarding η_2, i.e., regarding the law of closure.

(*) It suffices to state, that the distance OB_2 is independent of the location of C_2 on the horizontal.

That portion of the graph (Fig. 42) relating to C_2'' and γ_2'' illustrates the case when ζ_2^2 is imaginary, that is, the case for which the absissa $\rho\eta_2''$ of C_2'' is sufficiently small so that the circle γ_2'' is located entirely below the horizontal through D_1.

In an analogous manner we could establish the general conditions for which ζ_3 or ζ_4... etc. become zero, negative or imaginary; it is believed, however that no practical purpose is served in further treating this subject, and we will now pass to the discussion of some typical problems.

§ 20. — Counterblow of depression following complete closure.

In this § we will try to establish the intensity of the pressure of the counterblow which occurs at the intant $\theta + \mu$, i.e. at one phase interval counted from the instant of the complete closure.

First we will establish quickly the value of this pressure for gate operations terminating in the 1st, 2nd, 3rd phases, so as to be able to draw general conclusions and, as an immediate consequence, to plot the corresponding synoptical representation. The reader will remember, conforming to the notations established in § 18, and, because here we assume a well determined perturbing gate operation, that the symbol ζ_* (ζ_*^2 = pressure at the stoppage of gate operation) must be replaced by the symbol ζ_i, where i is the index of the phase of the operation during which the stoppage occurs; the symbols ζ_1, ζ_2..., per contra, must be replaced by ζ_{i+1}, ζ_{i+2}... etc.; regarding the symbols $\zeta_{i-1}, \zeta_{i-2}, \zeta_{i-3}$ etc., they correspond to the pressures which occur 1, 2, ... etc. phases before the closure, that is, to those related to the degrees of opening $\eta_{i-1} = \frac{1}{\theta}, \eta_{i-2} = \frac{2}{\theta}$..., as it is easy to verify.

CLOSURE IN THE FIRST PHASE.

$$0 < \theta < 1$$

The first two equations of the fundamental system, because $\eta_1 = \eta_2 = 0$, give

$$\zeta_1^2 - 1 = 2\rho$$
$$\zeta_1^2 + \zeta_2^2 - 2 = \theta \quad ;$$

in order that the pressure of the counterblow ζ_2^2 should reach a given value (naturaly < 1), it must be that

$$\rho = \frac{1}{2}(1 - \zeta_2^2), \quad \text{or:} \quad \zeta_2^2 = 1 - 2\rho, \qquad (65)$$

which results by eliminating ζ_1^2, and gives, in the cartesian synopsis, the loci of conduits situated in the zone of sudden closure $0 < \theta < 1$, for which the pressure of counterblow following the complete closure has a value given in advance

Inasmuch as equation (65) is independent of θ, these loci are segments of straight vertical lines, limited by the lines $\theta = 0$ and $\theta = 1$, and determined by the following table:

$\zeta_2^2 = +0,8$	$+0,6$	$+0,4$	$+0,2$	0	$-0,5$	$-1,0$	$-2,0$
$\rho =$ 0,1	0,2	0,3	0,4	0,5	0,75	1,0	1,5

which values were used in drawing these segments in the synopsis fig. 47 (*).

The value $\rho = 0.5$, therefore, is the limiting value of the characteristic for which the pressure of sudden closure, double of the pressure of regimen, is follewed by a counterblow which makes the pressure equal to zero. This phenomenon occurs, therefore, when the kinetic energy and the potential energy of the pipe have the same value. (§ 3, Note I).

CLOSURE IN THE SECOND PHASE.

$$1 < \theta < 2$$

Because $\eta_1 = \frac{1}{\theta}, \eta_2 = \eta_3 = 0$, the first 3 fundamental equations give

$$\zeta_1^2 - 1 = 2\rho\left(1 - \frac{\zeta_1}{\theta}\right)$$

$$\zeta_1^2 + \zeta_2^2 - 2 = 2\rho\frac{\zeta_1}{\theta}$$

$$\zeta_2^2 + \zeta_3^2 - 2 = 0$$

Eliminating ζ_1 and ζ_2:

$$\theta = \frac{2\rho\sqrt{2(1+2\rho+\zeta_3^2)}}{1+2\rho-\zeta_3^2} \qquad (66)$$

by means of which, assigning predetermined values to the pressure of the counter blow ζ_3^2, the loci of the conduits, for which these values will materialize in the zone $1 < \theta < 2$ can be found.

Assuming, for example, $\zeta_3^2 = 0$, equation (66) becomes

$$\theta = 2\rho\sqrt{\frac{2}{1+\rho}}, \quad \text{or:} \quad \rho = \frac{\theta}{8}(\sqrt{\theta^2+8}+\theta),$$

which gives the following pair of values:

θ = 1,0	1,2	1,4	1,6	1,8	2,0
ρ = 0,5	0,641	0,798	0,970	1,159	1,366

by means of which the locus $\zeta_3^2 = 0$ was drawn in fig. 47; this locus, as can be seen, is an arc with a slight concavity toward the right.

The loci corresponding to values between 0 and 1 of the counterblow ζ_3^2, have an analogous form; they meet the line $\theta = 1$ at points defined by equation (65) because equation (66) takes the form of (65) if we make $\theta = 1$; moreover, these loci meet the line $\theta = 2$ at points determined by

$$\rho^3 - \frac{1-\zeta_3^2}{2}\rho^2 - (1-\zeta_3^2)\rho - \left(\frac{1-\zeta_3^2}{2}\right)^2 = 0, \qquad (67)$$

obtained by making $\theta = 2$ in equation (66).

(*) We have plotted here negative counterblows also; of course, the figures have a physical meaning only if the pressures which they represent are within the limits of the atmospheric pressure.

Nevertheless, if such figures would indicate a depression below the atmospheric limit, they would give an idea of the intensity of perturbances resulting from the corresponding discontinuities of motion.

Assigning certain chosen values to ζ_3^2, equation (67) will furnish the corresponding values of ρ.

$\zeta_3^2 =$ 0,8	0,6	0,4	0,2	0,0	— 0,5	— 1,0	— 2,0	— 3,0
$\rho =$ 0,52	0,779	0,995	1,187	1,366	1,774	2,148	2,837	3,48

This table furnishes the points of the loci $\zeta_3^2 =$ const. for region $1 < \theta < 2$ (fig. 47).

In fig. 45, we have graphically demonstrated, by means of the circular diagram, a case very close to $\rho = 1.366$ $\zeta_3^2 = 0$; this diagram, in fact, was drawn for $\rho = 1.300$, and resulted in $\zeta_3 = 0.28$ or $\zeta_3^2 = 0{,}078$, which value is little different from 0.

CLOSURE IN THE THIRD PHASE

$2 < \theta < 3.$

The four first fundamental equations give, if we make

$$\eta_1 = \frac{2}{\theta}\,, \eta_2 = \frac{1}{\theta}\,, \eta_3 = \eta_4 = 0,$$

$$\zeta_1^2 - 1 = 2\rho\left(1 - 2\frac{\zeta_1}{\theta}\right)$$

$$\zeta_1^2 + \zeta_2^2 - 2 = 2\rho\left(2\frac{\zeta_1}{\theta} - \frac{\zeta_2}{\theta}\right) \tag{68}$$

$$\zeta_2^2 + \zeta_3^2 - 2 = 2\rho\frac{\zeta_2}{\theta}$$

$$\zeta_3^2 + \zeta_4^2 - 2 = 0.$$

By the same process as that used above, these equations will furnish the loci of the synopsis for which ζ_4^2 reaches a chosen, predetermined value; for this purpose ζ_3, ζ_2 and ζ_1, should be eliminated between the preceding equations.

We will limit this demonstration to the case where the pressure of the counterblow equals zero. Making $\zeta_4^2 = 0$ in the fourth equation of (68), the third will give

$$\zeta_2 = 2\frac{\rho}{\theta}\,,$$

this, substituted in the second, and after eliminating ζ_1 between this and the first equation, finally gives

$$20.\left(\frac{\rho}{\theta}\right)^4 - 2\left(\frac{\rho}{\theta}\right)^3 - 7\left(\frac{\rho}{\theta}\right)^2 + \left(\frac{2\rho - 1}{4}\right)^2 = 0\,, \tag{69}$$

which can be put in the form

$$\theta^4 - 2\,\frac{16\,\rho^3 + 56\,\rho^2}{(2\rho - 1)^2}\,.\,\theta^2 + \frac{320\,.\,\rho^4}{(2\rho - 1)^2} = 0\,, \tag{70}$$

a quadratic in θ^2, making it easier to solve numerically.

This equation, in the zone $2 < \theta < 3$ determines the locus of the conduits for which the pressure of the counterblow is equal zero; this locus passes through the points

$\theta =$	2,000	2,313	2,580	2,849	3,000
$\rho =$	1,366	1,600	1,800	2,000	2,112

by which it was plotted in fig 47.

This locus has sensibly the form of a straight line passing through the origin; it is easy to verify that this propriety is generally common to all loci $\zeta_4^2 =$ const., in the zone $2 < \theta < 3$.

CLOSURE IN THE i[th] PHASE.

It is not difficult to understand the reason that the loci $\zeta_4^2 =$ const., in the zone $2 < \theta < 3$, have the propriety just stated, and which is even more accentuated in the zone $3 < \theta < 4$, etc

Referring to § 10 of Note II, and remembering that, if ρ and θ increase, the maximum pressure of closure tends toward the value ζ_m, it naturally follows that the value of the pressure of the counterblow, at the instant one phase after closure, must tend toward the limiting value ζ_c^2 given by

$$\zeta_m^2 + \zeta_c^2 - 2 = 0,$$

or

$$\zeta_c^2 + \left(\left(\sqrt{\left(\frac{\rho}{2\theta}\right)^2 + 1} + \frac{\rho}{2\theta}\right)^2 - 2 = 0\right)$$

from which we have

$$\frac{\rho}{\theta} = (2 - \zeta_c^2)^{\frac{1}{2}} - (2 - \zeta_c^2)^{-\frac{1}{2}}. \tag{71}$$

We conclude that the loci of the conduits for which the pressure of the first counterblow succeeding closure reaches a given value ζ_c^2, have as a limiting form the plot of straight lines passing through the origin and represented by equation (71) which furnishes the following solutions:

$\zeta_c^2 = 0{,}8$	0,6	0,4	0,2	0,0	$-0{,}5$	$-1{,}0$	$-2{,}0$
$\frac{\rho}{\theta} = 0{,}182$	0,338	0,465	0,596	0,707	0,949	1,154	1,500

In order to give an idea of the approximation of the synoptical representation of this equation, we should note that the line $\zeta_c^2 = 0$ of the plot (71) cuts the line $\theta = 3$ at $\rho = 2.121$, very near to point $\rho = 2.112$ where the locus $\zeta_4^2 = 0$ of the zone $2 < \theta < 3$ cuts the same line $\theta = 3$.

It is, therefore, justified to adopt, without serious error, as the extremities of the loci $\zeta_4^2 =$ const. on $\theta = 3$, the points determined by the lines corresponding to equation (71), viz:

$\zeta_c^2 = 0{,}8$	0,6	0,4	0,2	0,0	$-0{,}5$	$-1{,}0$	$-2{,}0$
$\rho = 0{,}546$	1,104	1,395	1,788	2,212	2,847	3,462	4,500

Drawing, moreover, the straight lines joining these points with the corresponding points

$\rho = 0{,}520$	0,779	0,995	1,187	1,366	1,774	2,158	2,837

which are, as seen before, the intersections of the loci $\zeta_3^2 =$ const. of the zone $1 < \theta < 2$ (represented by (66)) with the lime $\theta = 2$, we have completed the synoptic representation in the zone $2 < \theta < 3$.

Beginning at $\theta > 3$, per contra, we will adopt, without further study, the straight lines of the plot of (71) as the points of the loci representing the conduits for which $\zeta_c^2 = \text{const.}$ The synopsis, fig. 47, is then completed in its full extent.

In the same manner, it vould be possible to study the cases of counterblows which follow the stoppage of the gate after a partial closure of the orifice; the reader would then find that in the case of partial closing, one can also realize zero or imaginary pressures.

It would be easy to establish cartesian synopsis illustrating such results; however, it is thought that this study should not be further extended here, especially as it was discussed in part (for a closing operation of duration, $< 2\mu$) in the last part of § 19.

§ 21. Counterblows of superpressure following opening operations.

The study of counterblows of superpressure following an opening operation presents technically interesting features so far they may be greater than 1 and may start dangerous pressures.

We have seen, in Note III, that the pressure, during an opening operation remains < 1; on the other hand, in § 18 of this Note we have stated in discussing the equation system (63) that, if, at the instant of gate stoppage, $\zeta_* < 1$, we will have, in the following rhytm $\zeta_1 > 1$, should the condition

$$2\rho_* - (\zeta_* + 1) < 0,$$

be satisfied. It follows that the phenomenon of superpressure of the counterblow > 1 can only be materialized, following an opening operation, if $\rho_* < 1$,

We will now find the value that this counterblow > 1 can reach.

This value is evidently given by the first equation of the system (61), viz..

$$\zeta_*^2 + \zeta_1^2 - 2 = 2\rho_* (\zeta_* - \zeta_1);$$

from which

$$\zeta_1^2 + 2\rho_* \zeta_1 - 2 + (2\rho_* \zeta_* - \zeta_*^2) = 0$$

and it is clear that ζ_1 will be the greater, the greater is the known term

$$2 + 2\rho_* \zeta_* - \zeta_*^2 .$$

It is easy to verify that this term will be a maximum when $\zeta_* = \rho_*$, and, in this case, the first equation of (61) becomes

$$\zeta_1^2 + 2\rho_* \zeta_1 - (2 + \rho_*^2) = 0,$$

or solving for ζ_1,

$$\zeta_1 = \sqrt{2\rho_*^2 + 2} - \rho_* \qquad (72)$$

from which it follows that to very small values of ρ_* correspond very large values of the counterblow ζ_1^2 (for instance for $\rho_* = 0.10$ we have $\zeta_1^2 = 1.745$).

But the assumption conducting to equation (72), in the case where ρ_* reaches very small values, is hydro-dynamically impossible. It is, in fact, entirely impossible to execute a gate operation which results in very small values of ζ_* for very small values of ρ_*; the reader is referred, on this subject, to the results of § 17.

Even in the least favorable case of a sudden opening for placing in service, equation (62 bis) of § 17 vould give

$$\zeta_* = \sqrt{\rho_*^2 + 1} - \rho_* \qquad (62)\ bis$$

which can conduct to $\zeta_* = \rho_*$ only if

$$\zeta_* = \rho_* = \frac{1}{\sqrt{3}} = 0{,}577.$$

but equation (72) furnishes, in this case, for the value of the counterblow

$$\zeta_1 = \sqrt{2 \times 0{,}577^2 + 2} - 0{,}577 = 1{,}056 \text{ and } \zeta_1^2 = 1{,}115 \qquad (73)$$

This value of counterblow, however, is not the maximum which can occur following a sudden opening. The maximum materializes for a much smaller value of ζ_*.

COUNTERBLOW FOLLOWING A SUDDEN GATE OPENING.

We will thoroughly study this case of the counterblow; it is the most interesting of all which can occur following an opening operation.

Introducing the value of the pressure of sudden opening;

$$\zeta_* = \sqrt{\rho_*^2 + 1} - \rho_*$$

into the equation giving the pressure of the counterblow

$$\zeta_*^2 + \zeta_1^2 - 2 = 2\rho_*(\zeta_* - \zeta_1)$$

we get

$$\zeta_1 = \sqrt{4\rho_*\sqrt{\rho_*^2 + 1} - 3\rho_*^2 + 1} - \rho_* \qquad (74)$$

which gives the figures of the following table:

$\rho_* = 0.1$	0.15	0.20	0.25	0.30	0.40	0.50	0.75	1.00
$\zeta_1 = 1.074$	1.087	1.102	1.1077	1.1081	1.098	1.077	1.000	0.912
$\zeta_1^2 = 1.153$	1.181	1.214	1.227	1.228	1.206	1.160	1.000	0.832

The counterblow of superpressure following a sudden gate opening, therefore, is maximum (*) for a value ρ_* laying between 0,25 and 0,30, but its relative magnitude is only little more than 20 % above normal; in all other cases it is less than this figure and, when $\rho_* > 0{,}75$, it is less than 1 or less than normal.

(*) Differentiating equation (74), we obtain

$$168\,\rho_*^6 + 225\,\rho_*^4 + 94\,\rho_*^2 = 9, \text{ which gives}$$

$$\rho_* = 0{,}28, \quad \text{and } \zeta_1 = 1.1085, \quad \zeta_1^2 = 1.229$$

It appears, therefore, contrary to general belief, that there is no danger of superpressure when a conduit is suddenly thrown into service. Only for high heads ($\rho < 0{,}75$, or $y_0 > 100\ m$) is such a gate operation followed by an oscillation of pressure which may result in a superpressure of 23 %; for medium and low heads the pressure tends asymptotically and without oscillation toward its value at regimen.

OBSERVATION

A sudden gate opening, followed by a sudden closure in the second phase would, however, be dangerous; the pressure which would be produced at the instant of closure, would be given, in this case by the two equations

$$\zeta_*^2 - 1 = -2\rho_*\zeta_*$$
$$\zeta_*^2 + \zeta_1^2 - 2 = +2\rho_*\zeta_*$$

from which

$$\zeta_1^2 = 1 + 4\rho_*\sqrt{\rho_*^2 + 1} - 4\rho_*^2. \qquad (75)$$

This relation, for $\rho_* < 0{,}75$, gives a greater pressure than that of sudden closure $1 + 2\rho$, while if ρ_* increases beyond 0.75, ζ_1^2 tends toward the value 3; we have, in fact:

for ρ_*	= 0.10	0.30	0.50	0.75	1.00	2.00	5.00	10.00
from (75), ζ_1^2	= 1.863	1.863	2.236	2.500	2.656	2.888	2.980	3.000
and $1 + 2\rho$	= 1.200	1.600	2.000	2.500	3.000	5.000	11.000	21.000

which confirm the preceding statement.

This simple example can serve as an illustration of the method to be used in studying the complex operations resulting from consecutive opening and closing motions of the gate; per contra, concerning the repeated rythmic operation which give occasion to the phenomenon called « resonance », the reader is referred to the following Note V.

It should be noted, finally, that the counterblows, which follow the opening operation of a duration longer than a rhythm, result in phenomena notably less in intensity; the reader can easily verify this fact by working out numerical examples.

The subject matter of this § can also be synoptically represented, without difficulty, by the reader himself.

NOTE V.

PHENOMENA OF RESONANCE.

Preliminary observations.

A conduit in service constitutes a system of elastic masses and members supported on fixed points; this elastic system, when dynamically impelled, reacts by periodic oscillations, the semi period of which is equal μ.

We can assume, therefore, that rhythmical impulses of a period 2μ will bring forth, in the conduit, phenomena called resonance, under the effect of which the pressure, due to the growing amplitude of successive variations, may reach values more or less greater than those corresponding to ordinary closing or opening operations executed according to a continuous law.

It also can be forseen, that, because these phenomena rest upon the alternative play of the transformation of kinetic energy into potential energy and vice versa, their intensity should be relatively greater for conduits having a large capacity for storing potential energy, that is, for conduits characterized by rather small values of ρ and which, consequently, are designed for high heads (see Note I § 3).

The gate operations which may result in rhythmical impulses able to produce resonances are the following:

A) Closing and opening operations, rhythmically following each other without stop; the resulting hydrodynamic phenomena we will designate by « Resonances due to rhythmical alternating operations. »

B) The progressive closing or opening operations executed in rhythmical steps; this closing operation is considerably interesting from a technical point of view; we will designate the phenomena resulting from this type of gate eperation by « Resonances due to rhythmical stepwise closure or opening. »

This Note, therefore, will be divided into two distinct parts in accordance with these two classes of operations.

FIRST PART.

RESONANCES DUE TO ALTERNATING RHYTHMICAL GATE OPERATIONS.

§ 22. — Circular and cartesian diagrams of the limiting pressures of resonance due to rythmical alternating operations.

Putting, in the equations of the fundamental system:

$$\eta_1 = \eta_3 = \eta_5 \ldots \text{ etc., } 1 = \eta_0 = \eta_2 = \eta_4 \ldots \text{ etc.}$$

in other words, if we assume that the gate alternately closes and opens the orifice partially, (due, for instance, to the « fanning » action of the governor in synchronism with the period of the conduit), the pressure will oscillate with an amplitude which may increase and may reach dangerous values.

It could not be stated, a priori, that the amplitude of these oscillations tends necessarily toward a limiting value; at first inspection, the study of this problem may even seem considerably complicated. However, the circular diagram of the interlocked series solves it in a surprisingly simple and elegant manner.

In fact, each of the two series of circles γ_i with odd and even indices respectively, result, on the assumption of alternate rhythmical gate operation, in a single circle: the circle γ_1 of center C_1, (coordinates $+\rho$ and $-\rho\,\eta_1$) (see figs. 48 & 49) for the series of odd indices, and the circle γ_2 of center C_2 (coordinates $-\rho$ and $+\rho\,\eta_1$) for the series of even indices. The two centers C_1 and C_2 and the origin O are on the same straight line and the two circles γ_1 and γ_2 pass through the point K situated at a distance $OK = \sqrt{2}$ on the line drawn from O at right angle to $C_1\,C_2$.

Let us determine, by the known graphic process, the interlocked series $\zeta_1, \zeta_2, \zeta_3 \ldots$ etc; it will be seen clearly (figs. 48 & 49) that the extremities of the segments $\zeta_1, \zeta_2, \zeta_3 \ldots$ etc, determine a broken line consisting of rectilinear orthogonal segments located between the two circles γ_1 and γ_2; it evidently results from this fact:

that the series of values with odd indices $\zeta_1\, \zeta_3\, \zeta_5 \ldots$ etc., tends toward the value Z_1 ordinate of the point K, while the series of values with even indices, $\zeta_2, \zeta_4, \zeta_6 \ldots$ etc., tends toward the value Z_2. absissa of point K.

The problem of the resonance due to an alternating rhythmical operation is, therefore, solved (*).

The limiting values Z_1 and Z_2, coordinates of K, moreover, have the important propriety of being independent from the characteristic ρ of the conduit.

(*) This will also prove the statement made in Note III, that the single circle γ_1 of the circular diagram of the interlocked series furnishes, at the same time, the pressure of the direct blow ζ_1^2 for opening or closure, the limiting pressure of the continued opening or closure ζm^2, and also the limiting pressures Z_1^2 and Z_2^2 of the resonance due to an alternating rhythmical gate operation. (See fig. 30 of § 15, Note III).

We have, in fact (fig. 48 and 49)

$$\frac{Z_2}{Z_1} = \frac{\rho\,\eta_1}{\rho} = \eta_1$$

or

$$\eta_1 Z_1 - Z_2 = O$$

and, besides, (76)

$$Z_1^2 + Z_2^2 - 2 = O$$

from which equations, we have for the limiting pressures Z_1^2 and Z_2^2

for $\eta_1 = 1 - \frac{1}{\theta}$	for $\eta_1 = 1 + \frac{1}{\theta}$	
Fig. 48	Fig. 49	
$Z_1^2 = \frac{2\,\theta^2}{\theta^2 + (\theta - 1)^2}$	$\frac{2\theta^2}{\theta^2 + (\theta + 1)^2}$	(77)
$Z_2^2 = \frac{2\,(\theta - 1)^2}{\theta^2 + (\theta - 1)^2}$	$\frac{2\,(\theta + 1)^2}{\theta^2 + (\theta + 1)^2}$	

It should be noted that fig. 48 corresponds to the case where the gate operation commenced with a closing motion, and we have, on this assumption,

$$Z_1 > Z_2;$$

fig. 49, on the contrary, corresponds to the case where the gate operation commenced with an opening motion, and we have, in this case

$$Z_1 < Z_2.$$

It is evident, however, that, from the point of view of limiting pressures, the case, of fig. 49 is only a specific case of that illustrated in fig. 48; it suffices to diminish by one unit the value of θ and to give ρ the value $\rho\,\eta_1$. In order to eliminate all confusion, in the following we will consider the assumption of fig. 48; we will assume, therefore, that the alternate operation has commenced with a closing motion in such a manner that Z_1^2 and Z_2^2 will be respectively the highest and lowest limiting pressure of resonance, i. e.,

Z_1^2, the limiting value of the upper series $\zeta_1^2, \zeta_3^2, \zeta_5^2 \ldots$

Z_2^2, » » » » » lower » $\zeta_2^2, \zeta_4^2, \zeta_6^2 \ldots$

Diagrams fig's 48 and 49, moreover, suggest the following important observation.

We have assumed that at the start of the gate operation a state of regimen prevailed aud the interlocked series was constructed in putting $\zeta_0 = 1$; this assumption, however, is not at all necessary.

We can give ζ_0 any value $\gtrless 1$, in other words, we can assume that a perturbing operation preceded the alternading rhythmic motion of the gate; fig's. 48

and 49 show that even for $\zeta_0 \gtreqless 1$, the series of the ζ of odd indices tend toward Z_1, and that of the ζ of even indices tend toward Z_2; the values of the successive members of the two series change, but the limits remain the same.

Thus, if the alternate rythmic operation was preceded by another operation which has reduced the orifice opening to $\eta_0 \gtreqless 1$ and has determined a pressure $\zeta_0^2 \gtrless 1$, our graphical procedure retains all its value; if suffices to make $\rho = \rho \eta_0$, i. e, to attribute to ρ a value which follows from the velocity of regimen corresponding to the degree of opening η_0.

These are, in their astonishing simplicity, the laws of resonance due to an alternating rhythmic gate operation. They even seem paradoxical in that the limiting values of the pressures of resonance are independent of the characteristic and therefore of the constructive elements of the conduit, and depend solely on the speed of operation. (*)

In order to keep the subject within bounds, we will only study the case of an alternating rhythmic operation executed in starting, from a stable regimen, by a closing motion. This is the case illustrated in fig. 48. We just saw that the case where the operation commences with an opening motion (fig. 49) can be reduced to the preceding case by a judicious selection of the parameter.

Fig. 48 shows that the series of the pressures with odd indices is an increasing one, i. e.,

$$\zeta_1 < \zeta_3 < \zeta_5 \ldots\ldots < Z_1,$$

while the series of even indices is decreasing, thus:

$$\zeta_2 > \zeta_4 > \zeta_6 > \ldots\ldots > Z_2.$$

This result depends evidently on the position of the centers C_1 and C_2, or, for a given θ, on the value of ρ, which, in fig. 48, is < 1 (the value of θ influences the inclination of the connecting line only).

If, on the contrary, we select for ρ a value sufficently greater than 1 (as shown on fig. 50) so that C_1 falls to right of the vertical drawn through K, and C_2 below the horizontal through K, it is clear that both series will be decreasing toward their respective limits.

The series of even indices, therefore, in both cases is a decreasing one. Moreover, it does not change into an increasing one for the intermediate case where ρ is so selected as to have C_1 fall to the right of the vertical passing through K, and C_2 below the horizontal through the same point. The two circles γ_1 and γ_2 have, in fact, for this assumption, the position represented by fig. 51, which results in an oscillation of decreasing amplitude of both the ζ's of even and odd indices, tending toward their respecting limiting values. Fig. 51 shows this very clearly.

The preceding reasoning is perfectly general in character, because the value θ has an influence only on the inclination of the line connecting the

(*) It should not be overlooked, however, that the speed of propagation and the length of the conduit L are not without influence upon the alternating gate operation, because they determine the period. It could not be assumed, for instance, that the conduit is rigid ($E = \infty$) and the water incompressible ($\varepsilon = \infty$) because then $a = \infty$ and $\mu = o$, and the closing operation should be executed in a time $= o$ or at an infinitely great speed.

the centers; therefore, if this line is fixed, the selection of ρ can only give occasion to the three cases of fig. 48, 50 and 51, viz:

1st case: (fig. 48), ρ is much < 1; the series $\zeta_1, \zeta_3 \dots$ is increassing, and the series $\zeta_2, \zeta_4 \dots$ is decreassing.

2nd case: (fig. 51), ρ has certain intermediate values; the two series each tend toward their respective limits in an oscillating manner, the amplitudes decreasing according to an assymptotic law.

3rd case (fig. 50) ρ is greater than a certain limit; the two series are decreasing.

Finally, if, as in the case of fig. 52, the value of ρ is so chosen that the vertical drawn through C_1, is at equal distance from the segments giving Z_1 and ζ_1, i. e., if,

$$2\rho = 1 + Z_2 \tag{78}$$

we will evidently have

$$\zeta_1 = \zeta_3 = \zeta_5 = \cdots = Z_1 \qquad \zeta_2 = \zeta_4 = \zeta_6 = \cdots = Z_2$$

We also could graphically investigate the limiting conditions for which the two preceding equalities would occur partially, that is, beginning with ζ_2, or ζ_3, or ζ_4 and complete in this manner the graphic theory of the phenomena of resonance due to an alternating gate operation, in constructing a synopsis of classification of the conduits from the point of view of these phenomena; this subject, however, lends itself better to an analytical study and will be so treated in the following paragraph.

CARTESIAN DIAGRAMS OF THE LIMITING PRESSURES OF RESONANCE.

Inasmuch as the pressures Z_1^2 and Z_2^2 are functions of the parameter θ only, it is clear that the loci of the conduits for which these pressures reach a given value are represented, in the cartesian synopsis, by horizontal straight lines: it is easy to determine the elements of a systematic diagram in calculating the following table from equation (77).

	$Z_1^2 = \dfrac{2\,\theta^2}{\theta^2 + (\theta - 1)^2}$	$Z_2^2 = \dfrac{2\,(\theta - 1)^2}{\theta^2 + (\theta - 1)^2}$
for θ = 1.000	= 2.00	= 0.00
= 1.190	= 1.95	= 0.05
= 1.295	= 1.90	= 0.10
= 1.500	= 1.80	= 0.20
= 1.724	= 1.70	= 0.30
= 2.000	= 1.60	= 0.40
= 2.366	= 1.50	= 0.50
= 2.896	= 1.40	= 0.60
= 3.757	= 1.30	= 0.70
= 5.450	= 1.20	= 0.80
= 10.475	= 1.10	= 0.90

It results from these values that the upper limit of the pressure can reach a maximum of twice the static head when the alternating operation is executed between the gate opening of the regimen and zero, i. e., complete closure. The circular diagrams give, in fact, for $\eta_1 = 0$, $Z_1 = \sqrt{2}$

The lower limit of the pressure becomes zero in this case; it can never become negative.

By means of the values of the preceding table it is easy to construct a single diagram of the limits Z_1^2 and Z_2^2; however, it is preferable to draw two distinct diagrams, in order to show the comparison between the maximum (or minimum) pressures of resonance due to an alternating gate operation and the maximum (or minimum) pressures due to an operation of closing or opening executed according to a continuous law.

The reader's attention is called to the fact that this is a first example of using the synopsis as a means of comparison of the effects of two or several types of gate operations.

DIAGRAMS OF THE Z_1^2 (fig. 53).

In the synopsis, fig. 53, the horizontals $\theta = const.$ are drawn, along which the upper limit of the pressures of resonance Z_1^2 has the values given in the preceding table; there is also plotted the diagram of the continuous closure of fig. 20 i.e., the plot of the hyperbolas (*) $\zeta_1^2 = const.$, and also that of the straight lines $\zeta_m^2 = const.$, ζ_1^2 and ζ_m^2 being respectively the pressure of the direct blow and the limiting pressure for a continuous gate closure (see Note II, § 12).

The horizontals $Z_1^2 = const.$ of the diagram of resonance intersect the corresponding hyperbolas $\zeta_1^2 = const.$ of the diagram of continuous closure, at the points of a curve z_1, the equation of which, naturally, is equation (78), or, substituting the value of Z_2:

$$\rho = \frac{1}{2}\left(\frac{(\theta - 1)\sqrt{2}}{\sqrt{\theta^2 + (\theta - 1)^2}} + 1\right) \qquad (79)$$

This curve, the locus of the conduits for which we have $Z_1^2 = \zeta_1^2$, passes through the point $(\theta = 1, \rho = 0{,}50)$; it has the asymptote $\rho = 1$, and lies entirely to the left of this ordinate.

The technically important phenomena of resonance due to alternating gate operation, therefore, are concentrated in the zone at the left of the locus z_1, in which (**) the upper limit of the resonance Z_1^2 is greater than the maximum pressure of closure.

We have drawn in full lines, the horizontals of the resonance diagram within the zone just mentioned, the arcs of the set of the hyperbolas between z_1 and s_1 and the set of straight lines $\zeta_m^2 = const.$ situated to the right of s_1. In this manner there was obtained a diagram of maximum pressures from the point of view of the two types of gate operations, viz.; the continuous closure and alternating closure and opening.

(*) The comparison, therefore, only applies to the maxima of the total rhythm, which, however, is justified by the fact that the law of resonance due to alternating gate operations does not admit of intermediate maxima, so that, without error, it is possible to omit from the zone of comparison the intermediate maxima of the continuous closure (see the observations at the end of the preceding paragraph).

(**) We have indicated, however, in fig. 53, the locus at which the horizontals of the resonance diagram intersect the homologous lines of the set $\zeta_m^2 = const.$, of the plot of continuous closure. But this locus, the equation of which can be easily determined, and which is also asymptotic to $\rho = 1$, has no technical importance.

In the zone to the left of z_1, the pressure of this latter is the maximum; in the zone between z_1 and s_1 the pressure of the direct blow of the continuous closure ζ_1^2 is the maximum, and in the zone to the right of s_1, we can assume, (als already observed in Note II) that the maximum is the limiting pressure of the continuous closure ζ_m^2.

If, in the zone of sudden closure ($\theta < 1$) we draw the vertical lines, being the loci of the conduits for which the pressure $1 + 2\rho$ has the same value as Z_1^2, and if these lines are projected to intersect the corresponding horizontals of the resonance diagram, a locus r is established (shown in dash and dotted line in fig. 53) the equation of which is

$$1 + 2\rho = Z_1^2,$$

$$\rho = \frac{1}{2} \cdot \frac{2\theta - 1}{\theta^2 + (\theta - 1)^2} \qquad (80)$$

This locus, therefore, passes through the point ($\theta = 1, \rho = 0{,}5$) and is asymptotic to the θ axis.

It has the propriety that for conduits located on this curve, the limiting pressure of resonance is equal to the pressure of sudden closure, while it is greater for conduits located between this curve and the θ axis (high and very high heads).

Thus, for conduits characterized by $\rho = 0.2$ and $\theta = 1.5$, which may correspond, for instance, to: $y_0 \cong 500$ to 700 m.; $v_0 \cong 2$ to $2{,}50$ m/sec.; $a \cong 1100$ m/sec.; $L \cong 1500$ m.; $\tau = 4$ to 5 sec., there will result: for the pressure of sudden closure $1 + 2\rho = 1.40$; for upper limit of the pressure of resonance due to an alternating gate operation $Z_1^2 = 1.85$.

Inasmuch as 3 or 4 alternating motions will suffice to nearly attain this limiting value (see fig. 48) the superpressure of resonance would be almost double of that of the sudden closure.

We have seen that the pressure Z_1^2 is constant no matter what is the value of ρ, that is, whatever the head; however, as we here deal with a relative value of the pressure, the effect of the resonance appears to be more dangerous, or at least, more imposing for the high than for the low heads, as it is capable, as shown in the preceding example, to induce superpressures of 400-600 m. and over.

We should observe, however, that, because the superpressure of resonance can not be greater than y_0, it is impossible that an alternating gate operation could result in a break of a conduit which is designed to withstand the static head with the usual coefficient of safety. We already stated, in § 9, that only defective conduits can burst.

DIAGRAMS OF THE Z_2^2 (fig. 54).

By a process analogous to the proceding, it is easy to construct a comparative diagram of the lower limiting pressures of resonance Z_2^2 and the minimum pressures of opening executed from a state of regime; the reader, to whom we recommend keeping in mind the remarks made at the beginning of this § regarding the first motion of the alternating gate operation, could, without difficulty construct this diagram and plot it on the synopsis of fig. 33. (*)

However, in order not to unduly lengthen this study, we shall limit oursetves to the most interesting case, i. e., to the opening operation for placing the conduit into service, (beginning with $\eta_0 = 0$, see § 17), and to compare the lower limiting 'pressure of resonance Z_2^2 with the minimum pressure of the direct blow ζ_1^2, which in this case, realizes the maximum depression obtainable by an opening operation.

With this object, we have drawn, in fig. 54, the horizontals $\theta = const$, the loci of the conduits for which the lower limit of resonance, Z_2^2, has the values of the preceding table, and we also plotted, on the same figure, the linear diagram of fig. 37, (Note III) which gives the pressures of the direct blow due to an opening for placing in service.

The horizontals $Z_2^2 =$ const., of the resonance diagram, intersect the straight lines corresponding to $\zeta_1^2 = const.$, of the diagram of opening for placing in service, along the curve the equation of which is easily found. According to the first equation of the system (56) of § 17, the pressure ζ_1^2 is given by

$$\zeta_1^2 - 1 = -2 \frac{\rho_*}{\theta_*} \zeta_1 \qquad (56)$$

in which ρ_* is the characteristic ot the conduit for the regimen corresponding to the degree of opening attained at the end of the opening operation of duration θ_*.

Keeping this definition in mind and, for simplicity, dropping the asterices, the proceding equation furnishes, for $\zeta_1^2 = Z_2^2$:

$$\rho = \theta \cdot \frac{1 - Z_2^2}{2 Z_2} = \frac{2}{2\sqrt{2}} \cdot \frac{\theta (2\theta - 1)}{(\theta - 1)\sqrt{\theta^2 + (\theta + 1)^2}} \qquad (81)$$

which is the equation of the locus f, which has for asymptotes $\theta = 1$ and $\rho = 0.50$ (fig. 54).

If, moreover, in the zone of opening, we draw the vertical lines representing the loci of the conduits for which the pressure of the sudden opening has the same relative value Z_2^2, and if these segments are projected to intersect the corresponding horizontals of the resonance diagram, a locus g is determined which, to its left, limits a zone including the conduits for which the maximum depression of resonance is greater than the depression of the sudden opening.

Putting $\theta = 1$ in the first equation of (56) and Z_2 for ζ_1, we obtain the equation of the curve g:

$$\rho = \frac{1 - Z_2^2}{Z_2} = \frac{1}{2\sqrt{2}} \cdot \frac{2\theta - 1}{(\theta - 1)\sqrt{\theta^2 + (\theta - 1)^2}} \qquad (82)$$

which curve has as assymptotes $\rho = o$ and $\theta = 1$ and which is shown by the dotted and dashed line in fig. 54.

(*) In order that such a diagram should completely correspond to that of fig. 53, which could be obtained in changing the sign of θ, it should be evidently assumed that the first motion of the alternating operation is (case of fig. 49) an opening motion. On this assumption Z_1^2 would become the lower limit and Z_2^2 the upper limit. In order to avoid confusion, we systematically adopted the hypothesis that the first motion is a closing motion, even though this assumption spoils the symmetry of the treatment of the subject.

§ 23. — Analytical studies of the laws of resonance Synopsis of classification.

The preceding has completely solved the problem of resonances due to alternating gate operation, at least from the point of view of the immediate technical interest, that is, regarding the maximum and minimum pressures of resonance.

The reader has seen that this most important part of the problem was solved by very simple means, which gave us not only the absolute value of these maxima and minima, but also their value compared to the maxima and minima produced by other methods of gate operation.

However, it will be found useful to complete this study of the laws of resonance by some analytical considerations which conduct to the summarized statement of these laws and to the establishment of a very elegant synopsis of classification.

For

$$\eta_1 = \eta_3 = \eta_5 \ldots . \qquad 1 = \eta_2 = \eta_4 \ldots .$$

we find, from the fundamental system, the equations:

$$\begin{aligned} \zeta_1^2 - 1 &= 2\rho(1 - \eta_1\zeta_1) \\ \zeta_1^2 + \zeta_2^2 - 2 &= 2\rho(\eta_1\zeta_1 - \zeta_2) \\ \zeta_2^2 + \zeta_3^2 - 2 &= 2\rho(\zeta_2 - \eta_1\zeta_3) \\ \zeta_3^2 + \zeta_4^2 - 2 &= 2\rho(\eta_1\zeta_3 - \zeta_4) \\ \text{etc.} & \qquad \text{etc.} \end{aligned} \tag{83}$$

while equations (76) result in

$$Z_1^2 + Z_2^2 - 2 = 0 \qquad \eta_1 Z_1 - Z_2 = 0 \tag{76}$$

Comparing equations (76) and (83) we find, without difficulty, the following analytical result; if the series $\zeta_1, \zeta_3, \zeta_5 \ldots$ and $\zeta_2, \zeta_4, \zeta_6 \ldots$ tend toward their respective limits Z_1 and Z_2, all members of equations (83) will separately tend toward zero.

The analytical demonstration of the laws of resonance is, therefore, accomplished if we prove that the numerical value (positive or negative) of the two members of equations (83) effectively tend toward zero; at the limit, equations (76) will then be verified. But this demonstration, while entirely elementary, and in which the fact must be brought out that both members of equations (83) can be $\gtreqless 0$, is lengthy and lacks elegance. It is easier to arrive at the result by subtracting the two equation (76) from the two members of each of equations (83) which gives:

$$\begin{aligned} \frac{\zeta_1 - Z_1}{1 - Z_2} &= \frac{2\rho - (1 + Z_2)}{2\rho\eta_1 + \zeta_1 + Z_1} \\ \frac{\zeta_2 - Z_2}{\zeta_1 - Z_1} &= \frac{2\rho\eta_1 - (\zeta_1 + Z_1)}{2\rho + \zeta_2 + Z_2} \\ \frac{\zeta_3 - Z_1}{\zeta_2 - Z_2} &= \frac{2\rho - (\zeta_2 + Z_2)}{2\rho\eta_1 + \zeta_2 + Z_1} \\ \text{etc.} & \qquad \text{etc.} \end{aligned} \tag{84}$$

Multiplying now, member by member, the first equation of this system (84) and the second, then the second and third, then the third and fourth, etc., it is easily established that

$$\frac{\zeta_2 - Z_2}{1 - Z_2} < 1 \quad \frac{\zeta_3 - Z_1}{\zeta_1 - Z_1} < 1 \quad \frac{\zeta_4 - Z_2}{\zeta_2 - Z_2} < 1 \quad \frac{\zeta_5 - Z_1}{\zeta_3 - Z_1} \ldots \text{etc.}$$

From the first, 3nd, 5ta... of these inequalities:

$$1 - Z_2 > \zeta_3 - Z_2 > \zeta_4 - Z_2 > \ldots\ldots > \zeta_{2}i - Z_2,$$

and from the 2nd, 4th, 6th

$$\zeta_1 - Z_1 > \zeta_3 - Z_1 > \zeta_5 - Z_1 > \ldots\ldots > \zeta_{2i+1} - Z_1$$

which demonstrates that the series $\zeta_2, \zeta_4, \zeta_6 \ldots\ldots$ and $\zeta_1, \zeta_3, \zeta_5 \ldots$ tend respectively toward the limits Z_2 and Z_1.

SYNOPSIS OF CLASSIFICATION.

We have seen in the preceding §, from elementary considerations of the circular diagrams, the manner in which the form of the law can be established, according to which the two series of ζ_{2i} and ζ_{2i+1} tend loward their respective limits.

For a given value of θ and for growing values of ρ we distinguish the following forms:

(1) Beginning at $\rho = 0$, and up to a certain value of $\rho\,(< 1)$, the series of the odd ζ's is increasing and the series of the even ζ's is decreasing.

(2) Subsequently, and up to a certain value of $\rho\,(> 1)$ the two series have an oscillatory character and tend toward their respective limits by oscillations of diminishing amplitude.

(3) Beyond this second limiting value of ρ, both series are decreasing.

We, therefore, can assume that the above three cases correspond to three principal regions of the synoptic plans, which we will denote by $\Omega_1, \Omega_2, \Omega_3$.

The finding of the limits to be attributed to these zones and the study of the manner of transition of the laws of resonance from one zone to the other will be the object of this §.

It is evident that the transition of the laws of region Ω_1, where $\zeta_{2i+1} < Z_1$ and $\zeta_{2i} > Z_2$, to the laws of region Ω_2, where ζ_{2i+1} and ζ_{2i} are alternately $\lesseqgtr$ than their respective limits, must occur in passing through the limiting case:

$$\zeta_{2i+1} = Z_1 \quad \text{or} \quad \zeta_{2i} = Z_2$$

where i may be $= 0, 1, 2, 3\ldots.$ etc. We can conclude that there exists between these two regions a set of limiting loci the equations of which can be obtained by making the numerators of equations (84) equal to zero; these new loci we will designate, in conformity with the notation z_1 adopted in fig. 53, by z_{2i+1}.

The existence of a set of limiting loci instead of one single limiting locus is not surprising if we consider that the transition of the laws of Ω_1 into the laws of Ω_2 may take place in a phase of the alternating operation of any order, which means that the form of the law of pressure of a given conduit subjected to an alternating operation, may obey the laws of resonance of Ω_1, for a certain number of phases, and beginning at the phase of rank i, may obey, to the contrary, the laws of resonance of Ω_2.

The same can be said regarding the transitions from Ω_2 to Ω_3. The only uncertainty remaining is to know how it is that the equations obtained in making the numerators of equations (84) zero can give both the limiting set between Ω_1 and Ω_2 and also that between Ω_2 and Ω_3. But this uncertainty disappears if we observe that the form of these equations is different depending on whether we deal with those of even indices or with those of odd indices. We will prove that the first are the limiting loci between Ω_1 and Ω_2, while the second are the limiting loci of Ω_2 and Ω_3.

THE SET z_{2i+1}

The equations of the loci of this set are obtained by equating to zero the numerators of the first, 3rd..... $(2i+1)^{th}$ equations of (84), which gives

$$\begin{array}{lll} 2\rho = 1 + Z_2 & \text{equation of } z_1 & \\ 2\rho = \zeta_2 + Z_2 & \text{»} \quad \text{»} \ z_3 & \\ 2\rho = \zeta_4 + Z_2 & \text{»} \quad \text{»} \ z_5 & (85) \\ \dots\dots\dots & & \\ 2\rho = \zeta_{2i} + Z_2 & \text{»} \quad \text{»} \ z_{2i+1} & \end{array}$$

where Z_2 must be replaced by its value determined from (77), and ζ_{2i} expressed by means of the system (83). These loci, z_{2i+1}, all have, as their common asymptote, $\rho = 1$, as it is evident that for $\theta = \infty$ we have

$$Lim\ \zeta_{2i} = Lim\ Z_2 = 1$$

The first of these loci, i. e., z_1, was already studied in the preceding § (see equation (79)), as the limiting locus of diagram 53; it intersects the line $\theta = 1$ at point $\rho = \frac{1}{2}$. We find, after eliminating ζ_2 from the 2nd equation of (85) by means of the first and 2nd of (83), and interposing the condition $\theta = 1$, i. e., $\eta_1 = 0$, that the locus z_3 intersects the line $\theta = 1$ at the point $\rho = \frac{1}{4}$.

In general, it can be demonstrated, tha the line $\theta = 1$ is intersected

by the loci	z_1	z_3	z_5	z_7	z_{2i+1}
at the points	$\rho = \frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{2i+2}$

These loci, therefore, are grouped in a set situated to the left of the asymptote $\rho = 1$.

It is equally easy to draw the limiting curve of this set for $i = \infty$, as, because ζ_{2i} tends toward the limit Z_2, the general equation (85) tends toward the limit

$$\rho = Z_2 = \frac{(\theta - 1)\sqrt{2}}{\sqrt{\theta^2 + (\theta - 1)^2}} \qquad (86)$$

which is the equation of the limiting curve sought; this curve passes through the points

$\theta = 1$	2	3	4
$\rho = 0$	0.632	0.784	0.848

and, evidently, has $\rho = 1$ as its asymptote.

The set of loci z_{2i+1}, is, therefore, situated between the curve determined by the first equation of (85) and that represented by equation (86). The curves can be very easily drawn as their equations give ρ explicitly as a function of θ (see fig. 55).

THE SET z_{2i}

The equations of the loci of this set are obtained by equating to zero the numerators of the 2nd, 4th ...... etc. equations of (84), which gives

$$\begin{aligned} 2\,\rho\,\eta_1 &= \zeta_1 + Z_1 \quad \text{equation of } z_2 \\ 2\,\rho\,\eta_1 &= \zeta_3 + Z_1 \quad \text{»} \quad \text{»} \; z_4 \\ 2\,\rho\,\eta_1 &= \zeta_4 + Z_1 \quad \text{»} \quad \text{»} \; z_6 \\ &\ldots\ldots\ldots\ldots \\ 2\,\rho\,\eta_1 &= \zeta_{2i-1} + Z_1 \quad \text{»} \quad \text{»} \; z_{2i} \end{aligned} \tag{87}$$

These loci have also, as their common vertical assymptote, $\rho = 1$, because we have evidently, for $\theta = \infty$

$$Lim\ \eta_1 = 1\,, \qquad Lim\ \zeta_{2i-1} = Lim\ Z_1 = 1\,.$$

Moreover, they have their common horizontal asymptote the line $\theta = 1$, because, on this assumption, $\theta = 1$:

$$Lim\ \eta_1 = 0 \qquad Lim\ \rho = Lim\ \frac{\zeta_{2i-1} + Z_1}{2\,\eta_1} = \infty$$

The loci Z_{2i}, therefore, have a hyperbolical form and tend toward the limit

$$\rho = \frac{Z_1}{\eta_1} = \frac{\theta^2 \sqrt{2}}{(\theta - 1)\sqrt{\theta^2 + (\theta - 1)^2}} \tag{88}$$

which is obtained in making: $i = \infty$, $Lim\ \zeta_{2i-1} = Z_1$.

The curve (88) has $\rho = 1$ and $\theta = 1$ for asymptotes and passes through the points

$\theta = 1.2$	1.5	1.7	2.0	2.0	3.0	4.0	5.0	10.0
$\rho = 9.73$	4.02	3.18	2.53	2.02	1.76	1.51	1.38	1.17

which points helped to draw it in fig. (55).

It is equally easy to draw the first curve of the set, Z_2 (fig. 55), because, in eliminating ζ_1 from the first equation of (87) by means of the first of (83), we obtain an equation which is only quadratic in ρ.

The set of the curves Z_{2i} is thus determined; it occupies, as can be seen in fig. 55, a very narrow zone, asymptotic to $\rho = 1$ and $\theta = 1$.

To complete the study of the laws of resonance of alternating gate operation in the three regions $\Omega_1, \Omega_2, \Omega_3$, we drew, in figures 56, 57, 58, the pressure diagrams for 3 conduits situated in the 3 zones; the ordinates of the summits of the curves were very accurately calculated by means of equation (83); these values are given in the following table:

Zone:	Ω_1		Ω_2		Ω_3	
Characteristics	$\theta = 2.5$	$\rho = 0.25$	$\theta = 1.5$	$\rho = 1.5$	$\theta = 3$	$\rho = 3$
$\zeta_1^2 =$	1.175		1.921		1.734	
$\zeta_2^2 =$		0.724		0.113		0.660
$\zeta_3^2 =$	1.353		1 622		1.460	
$\zeta_4^2 =$		0.607		0.226		0.622
$\zeta_5^2 =$	1.416		1.842		1.391	
$\zeta_6^2 =$		0.565		0.194		0 616
$\zeta_7^2 =$	1 450		1.789		1.3855	
$\zeta_8^2 =$		0.543		0.202		0.6155
	. . . .	. . . .	. . . . .	. . . . .	. .	. . . . . .
$Z_1^2 =$	1.471		1.800		1.385	
$Z_2^2 =$		0.529		0.200		0.615

The diagram, fig. 56, indicates the increasing series of pressures of odd indices and the decreasing series of the pressures of even indices.

The diagram fig. 57 illustrates clearly how the two series approach their respective limits by oscillations of decreasing amplitude. Fig. 58 shows the two decreasing series. (*)

The segments of these diagrams connecting the successive peaks and depressions were drawn in the form of straight lines; in fact, it is unnecessary to deal here with the intermediate maxima, which, as can easily be demonstrated, can not occur at the limiting values, which are the ones interesting from a technical point of view.

PART IIND

RESONANCE DUE TO FRACTIONAL PROGRESSIVE RHYTHMIC CLOSURE OR OPENING.

§ 24. — Fractional rhythmic closure. General formulas.

The present and following §'s give a complete treatise of the fractional, progressive, rhythmic closure which is a great deal more interesting than the analogous opening operation to be briefly mentioned in § 27.

(*) These series would be increasing if the operation had started with an opening gate movement; however, on the same assumption, diagrams 56 and 57 would not be subject to any change.

We consider a conduit (ρ, θ) being closed according to a law progressing by rhythmic fractions, that is, with intervals of stops of duration μ. In the case that θ is an integer

$$\theta = n$$

the operation is performed in a manner of diminishing the degree of opening by the same quantity $\frac{1}{\theta}$, during each phase of odd order, the 1st, 3rd, 5th etc.; and in having the vanes motionless during the phases of even order, the 2nd, 4th, 6th etc. It follows that the total duration of the operation up to complete closure will equal $2\theta - 1$ or $(2n - 1)\mu$ seconds.

In the case that θ is a fractional number between n and $n + 1$:

$$n < \theta < n + 1$$

we will assume, that the first motion has a duration of the fraction of the phase $(\theta - n)\mu$; the succeeding motions then will have, as before, a duration of a complete phase.

We, then, should introduce in the fundamental system (9) the following values of η:

$$\eta_1 = \eta_2 = \frac{n}{\theta}, \quad \eta_3 = \eta_4 = \frac{n-1}{\theta}, \ldots\ldots \tag{89}$$

$$\ldots\ldots \eta_{2n-3} = \eta_{2n-2} = \frac{2}{\theta}, \quad \eta_{2n-1} = \eta_{2n} = \frac{1}{\theta} \quad \eta_{2n+1} = 0$$

and obtain the set of equations:

$$\begin{aligned}
\zeta_1^2 \quad - 1 &= 2\frac{\rho}{\theta}(\theta - n\,\zeta_1) \\
\zeta_1^2 + \zeta_2^2 - 2 &= 2\frac{\rho}{\theta}(n\,\zeta_1 - n\,\zeta_2) \\
\zeta_2^2 + \zeta_3^2 - 2 &= 2\frac{\rho}{\theta}\Big(n\,\zeta_2 - (n-1)\,\zeta_3\Big) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\zeta^2_{2n-2} + \zeta^2_{2n-1} - 2 &= 2\frac{\rho}{\theta}(2\,\zeta_{2n-2} - \zeta_{2n-1}) \\
\zeta^2_{2n-1} + \zeta^2_{2n} - 2 &= 2\frac{\rho}{\theta}(\zeta_{2n-1} - \zeta_{2n}) \\
\zeta^2_{2n} + \zeta^2_{2n+1} - 2 &= 2\frac{\rho}{\theta}\zeta_{2n}
\end{aligned} \tag{90}$$

in the second members of which, when θ is an integer, we must substitute

$$n = \theta - 1$$

The series of the values of ζ_i satisfying the systems of equations which follow from (90) for divers chosen values of ρ and θ, within the limit of practical application, obey the following laws:

1st Law:

The ζ_i of even indices which satisfy equation (90) are little different from unity, thus:

$$\zeta_2 \backsim \zeta_4 \backsim \zeta_6 \ldots\ldots \zeta_{2n} \backsim 1$$

2nd Law:

The ζ_i of odd indices which satisfy equation (90) constitute a series of increasing values, thus:

$$1 < \zeta_1 < \zeta_3 < \ldots\ldots < \zeta_{2n+1}$$

Before discussing the elements of the demonstration, or rather, verification, of the above laws, the reader's attention is called to the important deduction which is their consequence and which justify these researches.

According to the 2nd law, the maximum pressure of fractional rhythmic closure naturally must be the last one of the series of odd indices, i. e., the pressure ζ_{2n+1} at the very instant of complete closure. If in conformity with the first law: $\zeta_{2n} \searrow 1$, the maximum pressure of fractional rhythmic closure, from the last equation of (90) can be written as

$$\zeta_{2n+1} \searrow 1 + 2\frac{\rho}{\theta} \tag{91}$$

which formula is highly interesting by its meaning and simplicity.

Inasmuch as the two laws enumerated conduct to the single formula (91), it is evident that a satisfactory demonstration of the laws will be given by a systematic proof that formula (91) can be numerically verified with sufficient accuracy within the synoptic field including those values of θ and ρ which enter into practical application; this demonstration will be the subject of the following §. But first, it is believed, that an illustration of the two preceding laws will be interesting and such illustration will be given by some graphical and analytical studies upon the series of values which satisfy the system (90).

CIRCULAR DIAGRAMS OF THE INTERLOCKED SERIES (Fig's. 59 and 60).

Applying the usual graphical analysis of § 6, Note 1, to the fractional rhythmic closing operation, the following remarks can be easily made on the resulting circular diagram:

A) The centers C_i of odd indices, C_1, C_3, C_5 ..... are alined upon a straight line inclined 45° to the axes and passing above the origin O, (as in the case of continuous closure). It follows that all circles γ_i with odd indices, γ_1, γ_3, γ_5 ..... intersect at the same point M of the bissectrix.

B) The centers C_i of even indices C_2, C_4, C_6 ..... are alined along the bissectrix of the exterior angle formed by the axes, and, consequently, all circles γ, wit even indices γ_2, γ_4, γ_6 ..... intersect at the same point K of the bissectrix, situated at a distance $=\sqrt{2}$ fron the origin, the coordinates of which are unity. Figures 59 and 60 clearly show that the points of intersection of both systems of circles, and the points which determine the exstremities of the vertical segments ζ_i (odd indices) and of the horizontal segments ζ_i (even indices) are all outside of the circle γ_0 of center O and radius $\sqrt{2}$ passing through K_0.

It follows that we always have

$$\zeta^2_{i-1} + \zeta^2_i > 2.$$

All first, and consequently all second members of the equations of the system (90), therefore, are positive quantities and the following inequalities result:

$$\theta > n\zeta_1 > n\zeta_2 > (n-1)\zeta_3 > (n-1)\zeta_4 > \ldots\ldots > 2\zeta_{2n-2} > \zeta_{2n-1} > \zeta_{2n}$$

which, written separately

$$\begin{aligned} &\frac{\theta}{n} > \zeta_1 > \zeta_2 > \frac{n-1}{n}\zeta_3 \\ &\zeta_3 > \zeta_4 > \frac{n-2}{n-1}\zeta_5 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\zeta_{2n-3} > \zeta_{2n-2} > \frac{1}{2}\zeta_{2n-1} \\ &\zeta_{2n-1} > \zeta_{2n} \end{aligned} \qquad (92)$$

The relations expressed by the system of inequalities (92) are the only ones susceptible of generalization and rigorous demostration for the fractional rhythmic closing operation which can not be subiected to the accurate and elegant synthesis by which we were able to examine the laws of resonances due to alternating operations.

DISCUSSION OF THE FIRST LAW.

Writing the first equation of (90) and the equations obtained by subtracting each equation (90) from the succeeding one, we arrive, after simplification, to the following system

$$\begin{aligned} &\zeta_1^2 - 1 = 2\frac{\rho}{\theta}(+\theta - n\zeta_1) \\ &\zeta_2^2 - 1 = 2\frac{\rho}{\theta}(-\theta + 2n\zeta_1 - n\zeta_2) \\ &\zeta_3^2 - 1 = 2\frac{\rho}{\theta}(+\theta - 2n\zeta_1 + 2n\zeta_2 - (n-1)\zeta_3) \\ &\zeta_4^2 - 1 = 2\frac{\rho}{\theta}(-\theta + 2n\zeta_1 - 2n\zeta_2 + 2(n-1)\zeta_3 - (n-1))\zeta_4 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \qquad (93)$$

where we can clearly see the formation of the successive polynomes of the second members.

Considering the second member of the second equation of (93), which can be written

$$2\rho\frac{n}{\theta}\left(-\frac{\theta}{n} + 2\zeta_1 - \zeta_2\right)$$

we can conclude that the trinome in parenthesis is a very small numerical quantity $\lesseqgtr 0$; it is composed, in fact, by a positive and 2 negative members, one of which is > and the other < than the half of the positive member, as shown by equation (92).

If, therefore, n is sufficiently large to make $\frac{\theta}{n}$ differ little from unity and if the value of ρ is not very important, the whole second member of the second equation of (93) is a very small numerical quantity, which justifies the statement of the first law

$$\zeta_2^2 - 1 \backsim 0 \qquad \zeta_2 \backsim 1 .$$

ζ_2, of course, may be $\gtrless 1$, and figures 59 and 60 illustrate the two cases.

Also, writing the second member of the 4th equation of (93) in the form

$$2\rho\frac{n}{\theta}\left(-\frac{\theta}{n} + 2\zeta_1 - \zeta_2 - \zeta_3 + 2\frac{n-1}{n}\zeta_3 - \frac{n-1}{n}\zeta_4\right)$$

we note that the first part of the polynome in parenthesis is equal to the trinome, the value of which was just discussed, and, therefore, is very small; the second part has the same propriety, one of the negative terms being >, the other < than one-half of the positive term, as shown by the two first equations of the system (92). It is, therefore, justified to state that

$$\zeta_4^2 - 1 \backsim 0 \qquad \zeta_4 \backsim 1$$

This procedure is evidently general in character; extended to ζ_{2n}, it permits us to affirm that the proposition of the first law

$$\zeta_2^2 - 1 \backsim 0 \qquad \zeta_{2n} \backsim 1$$

is very plausible. As to the study of the limits of its application, these must be subjected to numerical trials.

DISCUSSION OF THE SECOND LAW.

We already observed, that, as a conseguence of the first law, now established, $\zeta_{2n} \backsim 1$, the last equation of (90) enables us to write the espression (91) of the pressure at the instant of closure

$$\zeta_{2n+1} \cong 1 + 2\frac{\rho}{\theta} \qquad (91)$$

We also stated that this pressure is the greatest of the successive maxima of the pressures resulting from a fractional rhythmic closure, which obey the system of inequalities

$$\zeta_1 < \zeta_3 < \zeta_5 < \ldots\ldots < \zeta_{2n+1} \qquad (94)$$

This system, in fact, is verified within the limits of values of ρ and θ which occur in pratical applications.

Considering the circular diagrams fig. 59 and 60, it is clear that, ζ_2 being nearly equal 1, the two vertical segments which give ζ_1 and ζ_3 fall close to each other. As the arc γ_3 is drawn above arc γ_1, in view of the relative position of the centers C_1 and C_3, it results that

$$\zeta_3 > \zeta_1 .$$

This reasoning is of a general character and can be repeated for ζ_{2i-1} and ζ_{2i+1}, if ζ_{2i-2} and ζ_{2i} are sufficiently close to unity,

The inequalities (94) are not susceptible of a more rigorous demonstration; it is even possible to show (*), that some of them may not hold in certain zones of the synoptic field; but these zones are beyond the limits of practical applications, as we must exclude (as indicated in § 5 of Note 1) the assumption of small values of ρ combined with large values of θ, or vice versa, while it should be remarked that the numerical values of ρ encountered in practice are comprised within fairly narrow limits. In order to confirm that equation (91) holds with the best approximation precisely in that part of the synoptic field which is especially interesting from a practical point of view, we observe that the relation

$$\zeta_{2n} \backsim 1$$

can also be written,

$$\zeta_{2n} = 1 \pm \varepsilon$$

(*) To illustrate this statement, we will demonstrate that for certain values of ρ and θ the condition $\zeta_1 = \zeta_3$ may hold in contradiction to the first inequality of the system (94).

Let us put $\zeta_1 = \zeta_3$ in the 3rd equation of the system (90); then, comparing this equation with the 2nd of the same system, we obtain:

$$2n\,\zeta_2 = (2n-1)\,\zeta_1 .$$

Eliminating ζ_2 between this and the 2nd of (90) we obtain a quadratic in ζ_1, from wich, eliminating ζ_1 by neans of the lst equation of (90) we can write:

$$(\alpha) \qquad \frac{\theta}{n} = \frac{2\rho\sqrt{2n(8n^2-2n+1)(2\rho+4n+1)}}{2\rho(8n^2-4n+1)-4n+1}$$

Because we have $\quad n < \theta < n+1$

equation (α) must satisfy the condition:

$$(\beta) \qquad \frac{n+1}{n} > (\alpha) > 1$$

A thorough study of the compatibilities between the numerical values and signs involved shows that equation (β) can be satisfied only by values of ρ close to unity and large values of n. For example, it can not be satisfied for $n < 10$ but is for

$n =$	15	20	50
$\beta =$	1,0424	1,0317	1,0125

with which values, equation (α) gives

$\theta =$	15,900	20,898	50,944

and the verification of the fact that the choice of these values result in $\zeta_1 = \zeta_3$ can only be made in working to 5 decimals. It appears that the result has no practical value inasmuch as these values of ρ and θ are well beyond their normal limits. Finally, the relation $\zeta_1 = \zeta_3$ has no practical importance as not the first, but the last terms of the inequalities (94) are interesting.

designating by ε a very small quantity. Substituting this expression of ζ_{2n} in the last equation of the system (90), we obtain, neglecting ε^2

$$\zeta^2_{2n+1} = 1 + 2 \cdot \frac{\rho}{\theta} \pm 2\varepsilon \left(\frac{\rho}{\theta} - 1\right) \qquad (91\ bis)$$

This new form shows that equation (91) has the more approximation, the less the values of the parameters ρ and θ differ.

§ 25. — Numerical researches of the maximum pressures due to fractional rhythmic closure. Cartesian Synopsis.

The importance of the systematical research of maximum pressures due to fractional rhythmic closure (which will be called, for simplicity, fractional closure) resides in the fact that these pressures considerably surpass the maximum pressures due to a continuous closure executed with the same speed (same value of θ), because of the following propositions:

A. The pressure of the fractional closure is always greater than the pressure of the direct blow.

B. The pressure of the fractional closure is greater than the limiting pressure ζ^2_m, of the continuous closure (§ 10 Note II), for $\rho : \theta < 3:2$, that is, for almost the whole of the synoptic field which is of practical interest.

Assuming (excepting if the numerical results would prove the contrary) the value given by equation (91) for the pressure of the fractional closure, proposition A can be expressed by the inequality

$$\zeta_1^2 < 1 + 2\frac{\rho}{\theta}$$

in which ζ_1^2, the pressure of the direct blow, has the value given by the equation

$$\zeta_1^2 - 1 = 2\frac{\rho}{\theta}\left(\theta - (\theta - 1)\zeta_1\right)$$

Combining these two relations, we obtain

$$1 > \theta - (\theta - 1)\sqrt{1 + 2\frac{\rho}{\theta}}$$

from which we deduce

$$1 + 2\frac{\rho}{\theta} > 1$$

which, of course, always holds good.

Proposition A, moreover, is evidently included in the system of inequalities (94), which are partly confirmed by the above discussion.

According to statement B, the pressure of the fractional closure must also (between certain limits) be greater than the limiting pressure of continuous closure given by equ. (19) of § 10

$$\zeta^2_m - \frac{\rho}{\theta}\zeta_m - 1 = 0$$

which can be transposed in the inequality

$$1 + 2\frac{\rho}{\theta} > \left(\sqrt{\left(\frac{\rho}{2\theta}\right)^2 + 1} + \frac{\rho}{2\theta}\right)$$

which reduces easily to

$$\frac{\rho}{\theta} < \frac{3}{2} \qquad (95)$$

Proposition B therefore is also verified.

Moreover, for the conduits located on the line $\rho : \theta = 3 : 2$ of the synoptic plan, we have $\zeta^2_m = 1 + 2\frac{\rho}{\theta} = 4$, in other words, both the continuous and fractional closures result in a pressure 4 times that of the stable regimen. It is, therefore, evident that we are at the limit of the synoptic plan which is interesting from a practical point of view, especially from that of power installations.

We will, therefore, investigate, by numerical calculation, in order to verify the laws of fractional closure, the angular zone extending between the θ axis and the line $\rho : \theta = 3 : 2$, wich passes through the origin (fig. 61). For the remaining zone (which, as mentioned, is not practically interesting) the limiting pressure of the continuous closure, ζ^2_m, is larger than the pressure of the fractional closure.

In these calculations, we will select for the different successive values of θ, those of conduits represented by points lined up with respect to the origin, so that for a series of conduits we shall have $\rho : \theta = const.$, which will permit an easier comparison with the results given by equation (91).

Zone $1 < \theta < 2$.

Putting, in system (90), $n = 1$, it reduces to 3 equations:

$$\begin{aligned} \zeta_1^2 - 1 &= 2\frac{\rho}{\theta}(\theta - \zeta_1) \\ \zeta_2^1 + \zeta_3^2 - 2 &= 2\frac{\rho}{\theta}(\zeta_1 - \zeta_2) \qquad (96) \\ \zeta_2^2 + \zeta_3^2 - 2 &= 2\frac{\rho}{\theta}\zeta_2 \end{aligned}$$

Applying this system to determine ζ_3^2 for $\theta = 1{,}5$ and $\theta = 2$ and for diverse values of ρ within the limits of (95), so selected that $\rho : \theta = const.$, we obtain for $\theta = 1{,}5$

$\rho\ = 0{,}15$	0,375	0,75	1,128	1,50	1,875	2,25
$\zeta^2_2 = 1{,}267$	1,590	2,058	2,519	3,000	3,507	4,039

and for $\theta = 2$

$\rho\ = 0{,}20$	0,50	1,00	1,50	2,00	2,50	3,00
$\zeta_3^2 = 1{,}334$	1,681	2,119	2,540	2,997	3,500	4,048

which corresponds to

$\frac{\rho}{\theta} = 0{,}10$	0,25	0,50	0,75	1,00	1,25	1,50

by which there results from (91)

$1 + 2\frac{\rho}{\theta} = 1{,}20$	1,50	2,00	2,50	3,00	3,50	4,00

Comparing the series of values of $1 + 2\frac{\rho}{\theta}$ with the values obtained by the system (96) for $\theta = 1{,}5$ and $\theta = 2$, we find that the expression of the pressure of the fractional closure given by (91) is satisfactorily approximate as long as $\rho : \theta > 0{,}50$, while it is sensibly erroneous for values of $\rho : \theta < 0{,}50$; in which case the effective values of the pressure ζ_3^2 are greater than the approximate values derived from (91).

The approximation of (91) is best for $\rho : \theta = 1$, conforming to the observation made at the end of the preceding §.

Zone $2 < \theta < 3$.

Substituting $n = 2$ in the system (90), we have the following 5 equations:

$$\zeta_1^2 - 1 = 2\frac{\rho}{\theta}(\theta - 2\zeta_1)$$

$$\zeta_1^2 + \zeta_2^2 - 2 = 2\frac{\rho}{\theta}(2\zeta_1 - 2\zeta_2)$$

$$\zeta_2^2 + \zeta_3^2 - 2 = 2\frac{\rho}{\theta}(2\zeta_2 - \zeta_3) \tag{97}$$

$$\zeta_3^2 + \zeta_4^2 - 2 = 2\frac{\rho}{\rho}(\zeta_3 - \zeta_4)$$

$$\zeta_4^2 + \zeta_5^2 - 2 = 2\frac{\rho}{\theta}\,.\,\zeta_4$$

from which, limiting ourselves to the case $\theta = 3$, we obtain

for ρ	= 0,30	0,75	1,50	2,25	3,00	3,75	4,50
ζ_5^2	= 1,395	1,702	2,119	2,541	2,997	3,495	4,044

while from (91) we get

$1 + 2\frac{\rho}{\theta}$ = 1,20	1,50	2,00	2,50	3,00	3,50	4,00

This system of values is subject to the same observations as already made regarding the values derived from (96), except that the error of formula (91) for $\rho : \theta < 0,50$ is even more noticeable.

Zone $3 < \theta < 4$.

Substituting $n = 3$ in system (90) gives 7 equations from wich, limiting ourselves to the case $n = 4$ we obtain

for ρ	= 0,40	1,00	2,00	3,50	4,00	5,00	6,00
ζ_7^2	= 1,411	1,703	2,119	2,541	2,997	3,495	4,042

while from (91)

$1 + 2\frac{\rho}{\theta}$ = 1,20	1,50	2,00	2,50	3,00	3,50	4,00

showing that the error for small values of $\rho : \theta$ is increasing. In fact, for $\rho : \theta = 0,10$ the superpressure ζ_7^2 is almost the double of that given by equation (91).

An analogous study for the zone $4 < \theta < 5$ will confirm the above results.

CARTESIAN SYNOPSIS.

Let us gather the previous results into a single table. Observing that for $\theta = 1$, formula (91) and the first equation of (90) both give the pressure as $1 + 2\rho$, (because the operation then is a sudden closure) this table will be as follows:

	$\frac{\rho}{\theta}$ = 0.10	0.25	0.50	0.75	1.00	1.25	1.50
$\theta = 1$	ζ_1^2 = 1.200	1 500	2.000	2.500	3.000	3.500	4.000
$\theta = 1,5$	ζ_3^2 = 1.267	1.590	2.058	2.519	3.000	3.507	4.039
$\theta = 2$	ζ_3^2 = 1.334	1.681	2.119	2.540	2.997	3.500	4.048
$\theta = 3$	ζ_5^2 = 1.395	1.702	2.119	2.541	2.997	3.495	4.044
$\theta = 4$	ζ_7^2 = 1.411	1.703	2.119	2.541	2.997	3.495	4.042
$1 + 2\frac{\rho}{\theta}$	= 1.20	1.50	2.00	2.50	3.00	3.50	4.00

By making analogous calculations for intermediate values, we obtain, by convenient interpolation, the elements for the construction of the cartesian synopsis, (fig. 61), which gives an adequate representation of the law of maximum pressure induced by a fractional rhythmic closure.

Excepting the stated anomalies for small values of $\rho : \theta$, this law, therefore, presents itself as a generalization of the law of the pressure of sudden closure, so that

$\zeta_1^2 = 1 + 2\rho$ = the pressure of sudden closure can be considered a particular case of

$\zeta_{2n+1} = 1 + 2\frac{\rho}{\theta}$ = the pressure of the fractional closure.

This proposition, therefore, synthetizes two interesting categories of phenomena.

In (fig. 61) we have extended the synoptic representation to the line $\rho : \theta = 3 : 2$, in other words, for the whole zone in which

$$1 + 2\frac{\rho}{\theta} > \zeta^2{}_m{}_3^2$$

In reality, however, the zone should be restrained to that part of the synoptic plan where $1 + 2\frac{\rho}{\theta}$ is greater than the maximum pressure of continuous closure, which, in this region (see Note II, fig. 18) may be the pressure at the instant of the complete closure, which pressure is greater than the limiting pressure $\zeta^2{}_m$. For example, in the zone $1 < \theta < 2$, the final pressure of continuous closure is given by

$$\zeta_1^2 - 1 = 2\frac{\rho}{\theta}(\theta - \zeta_1)$$

$$\zeta_1^2 + \zeta_2^2 - 2 = 2\frac{\rho}{\theta}\zeta_1$$

Putting $\zeta_2^2 = 1 + 2\frac{\rho}{\theta}$, the second equation gives $\zeta_1 = 2\frac{\rho}{\theta} - 1$. This value, substituted for ζ_1 in the first equation, after some simplifications, results in

$$\rho = \frac{3 + \theta}{4} \cdot \theta \qquad (99)$$

which is the equation of the limiting locus sought.

From (99) we obtain

for	$\theta = 1$	1,5	2
	$\rho = 1$	1,69	2,50

The curve so plotted differs very little from the straight line joining the points ($\theta = 1$, $\rho = 1$) and ($\theta = 2$, $\rho = 2{,}5$).

In a similar way it can be demonstrated for the zone $2 < \theta < 3$, that the limiting locus along which the final pressure of continuous closure $\zeta_3^{\,7}$ is equal $1 + 2\frac{\rho}{\theta}$ is very closely the line joining the points ($\theta = 2$, $\rho = 2{,}5$) and ($\theta = 3$, $\rho = 4$), etc.

Finally, the true limit of the zone of the synopsis in which the pressure of the fractional closure is greater than the maximum pressure of the continuous closure, is not the line $\rho : \theta = 3 : 2$, but a parallel line drawn through the point $(\theta = 1, \rho = 1)$. This line is shown by dots and dashes in fig. 61.

The cartesien synopsis of the pressures of fractional closure is now sufficiently illustrated.

§ 26. — Comparison of the pressures of fractional rhythmic closure, the pressures of continuous closure and the pressures of resonance due to an alternating gate operation.

PRESSURES OF FRACTIONAL CLOSURE AND PRESSURES OF CONTINUOUS CLOSURE.

We have seen that in the angular zone located between the θ axis and the line $\rho : \theta = 3 : 2$, the pressure of fractional closure determined by equation (91) is greater than the limiting pressure of continuous closure ζ^2_m given by equation (19) of § 10:

$$\zeta^2_m - \frac{\rho}{\theta}\zeta_m - 1 = 0 \qquad (19)$$

that is

$$1 + 2\frac{\rho}{\theta} - \zeta^2_m < 0 \qquad (100)$$

Let us find, first, the numerical values of the difference

$$\left(1 + 2\frac{\rho}{\theta}\right) - \zeta^2_m$$

by using the same series of values of the ratio $\rho : \theta$ as that adopted in the preceding §.

$\frac{\rho}{\theta} =$	0,10	0,25	0,50	0,75	1,00	1,25	1,50
$1 + 2\frac{\rho}{\theta} =$	1,200	1,500	2,000	2,500	3,000	3,500	4,000
$\zeta^2_m =$	1,105	1,283	1,641	2,082	2,618	3,255	4,000
Difference =	0,095	0,217	0,359	0,418	0,382	0,245	0

The difference, therefore, is a maximum for $\rho : \theta$ about 0,75. In fact, making $\rho : \theta = x$, and differentiating the first member of (100) with respect to x, we obtain the condition that the difference $\left(1 + 2\frac{\rho}{\theta}\right) - \zeta^2_m$ be a maximum:

$$1 - \zeta_m \frac{d\zeta_m}{dx} = 0$$

while, differentiating (19), we obtain

$$(2\,\zeta_m - x)\,\frac{d\,\zeta_m}{d\,x} - \zeta_m = 0\,.$$

Eliminating $\frac{d\,\zeta_m}{d\,x}$ and replacing x by its value, we get

$$\zeta^2_m - 2\,\zeta_m + \frac{\rho}{\theta} = 0\,,$$

Finally, eliminating ζ_m by the aid of (19) we obtain the sought for condition:

$$\left(\frac{\rho}{\theta}\right)^3 - \left(\frac{\rho}{\theta}\right)^2 + 4\,\frac{\rho}{\theta} - 3 = 0\,. \qquad (101)$$

which is satisfied by $\rho : \theta = 0{,}783$, which confirms the preceding observation.
With this value of the ratio $\rho : \theta$ we obtain

$$1 + 2\,\frac{\rho}{\theta} - \zeta^2_m = 2{,}566 - 2{,}148 = 0{,}419$$

which differs very little from the value of this difference corresponding to $\rho : \theta = 0{,}75$.

This first study, therefore, shows that the pressure of fractional closure surpass the limiting pressure of continuous closure by an amount of more than 40 % of the static pressure. Finally, if we consider, instead of the approximate value given be (91), the effective value of the pressure of fractional closure, calculated for each particular case, and instead of the limiting pressure of continuous closure, the maximum effective pressure, we will observe that the differences between the corresponding pressures of the two series sensibly deviate from the values of the preceding table (although conserving the same order of magnitude), and this for two reasons:

1st) For small values of θ the maximum pressure of continuous closure is sensibly greater than the limiting pressure ζ^2_m given by (19).

2nd) For small values of ρ the pressure of fractional closure is notably superior (as we have already remarked) to the approximate value $1 + 2\,\frac{\rho}{\theta}$ derived from (91).

Making use of the synopsis of classification fig. 18, to determine in each case of continuous closure, that pressure of total rhythme which is the maximum pressure, i. e.,

$$\zeta_1^2 \text{ or } \zeta_2^2 \,\ldots\ldots\ldots\, \text{ or } \zeta^2_i$$

depending on whether the point representing the conduit falls in the zone

$$\Sigma_1 \text{ or } \Sigma_2 \,\ldots\ldots\ldots\, \text{ or } \Sigma_i$$

and calculating this pressure for the same cases as those examined in the preceding, we establish the following table:

COMPARATIVE TABLE OF MAXIMUM PRESSURES.

α) *Fractional closure.*
β) *Continuous closure.*

$\frac{\rho}{\theta}$	= 0.10	0.25	0.50	0.75	1.00	1.25	1.50
			(θ = 1.5)				
ρ	= 0.15	0.375	0.75	1.125	1.50	1 875	2.25
α)	= 1.267	1.590	2 058	2.519	3.000	3.507	4.039
β)	= 1.190	1 450	1.825	2.149	2.944	3.562	4.203
Difference	= 0.077	0 140	0.233	0 370	0.056	-0.055	-0.164
			(θ = 2)				
ρ	= 0.20	0.50	1.00	1.50	2.00	2.50	3.00
α)	= 1 334	1.681	2.119	2.540	2.997	3.500	4.048
β)	= 1.182	1.407	1.698	2.158	2.798	3.500	4.248
Difference	= 0.152	0.274	0.421	0.382	0.199	0.000	-0.200
			(θ = 3)				
ρ	= 0.30	0.75	1.50	2.25	3.00	3.75	4.50
α)	= 1.395	1.702	2.119	2.541	2.997	3 495	4.044
β)	= 1.168	1.341	1.656	2.108	2.691	3 377	4.188
Difference	= 0.227	0.361	0.463	0.433	0.306	0.118	-0.144
			(θ = 4)				
ρ	= 0.40	1.00	2.00	3.00	4.00	5.00	6.00
α)	= 1.411	1.703	2.119	2.541	2.997	3.495	4.042
β)	= 1.156	1.294	1.643	2.086	2.654	3.339	4.132
Difference	= 0.255	0.409	0.476	0.455	0.342	0.156	-0.092

This table confirms, therefore, that the effective difference between the pressure of fractional closure and the maximum pressure of continuous closure, may, for certain conduits, reach 40 to 50 % of the static pressure. Moreover, it indicates, that this difference may even increase for small values of ρ and increasing values of θ; for instance, in the third column ($\rho : \theta = 0{,}50$), we see that beyond $\rho = 1$, the pressure of fractional closure remains equal to 2.119, while the pressure of continuous closure diminishes with increasing values of θ. Consequently, the difference between the two pressures should increase.

Cases corresponding to values of θ greater than those examined are not possible in practice; the last values of the table already are outside of the field of probabilities.

The fractional rhythmic closure, in fact, can not be considered as an intentional gate operation; it will occur in practice only accidentally due to the action of the governor under the influence of entirely uncommon conditions, or to the involuntary mistake of an operator. It is hard to conceive that such accidental operation could retain its rhythmic form for more than 2 or 3 phases.

Only that part of the synoptic field, for which $\theta \leqq 2$ has, therefore, a practical importance, but it remains true, nevertheless, that the general results of these researches merit the whole attention of the engineer, inasmuch as the resonance due to a fractional closure, a phenomenon formerly ignored, gives for almost the whole of the synoptic field the maximum superpressure which can be realized by a given speed of gate operation. The exception is a small zone, where the alternating operation is the most dangerous.

Numerical examples.

1st) $(\theta = 2 \quad \rho = 0{,}20$

which can be realized for

$$y_0 = 500 \qquad v_0 = 2 \qquad a = 1080 \qquad L = 1620 \qquad \tau = 6''$$

Closing the gates halfway in 3 seconds, then leaving them stationary for 3 seconds, and finally closing them in 3 seconds, we obtain

a superpressure of fractional closure	$0{,}334 \times 500 = 167$ m
while the superpressure of continuous closure is	$0{,}182 \times 500 = 91$ m
difference in superpressure	76 m

2nd) $(\theta = 2 \quad \rho = 0{,}50)$

which can be realized for

$$y_0 = 200 \qquad v_0 = 2 \qquad a = 1000 \qquad L = 1250 \qquad \tau = 5''$$

With an analogous operation of intervals of 2,5″ each we obtain

a superpressure of fractional closure	$0{,}681 \times 200 = 136$ m
the superpressure of continuous closure is	$0{,}407 \times 200 = 81$ m
difference in superpressure	76 m

3rd) $(\theta = 3 \quad \rho = 1{,}50)$

which can be realized for

$$y_0 = 75 \qquad v_0 = 2{,}5 \qquad a = 800 \qquad L = 720 \qquad \tau = 5'',5$$

superpressure of fractional closure	$1{,}119 \times 75 = 84$ m
superpressuse of continuous closure	$0{,}656 \times 75 = 49$ m
difference	35 m

PRESSURES OF FRACTIONAL CLOSURE AND PRESSURES OF ALTERNATING GATE OPERATION.

We have seen that the pressure of fractional closure is always greater than the pressure of the direct blow, and we stated in § 22 (See figs. 53 and 54) that the pressure of resonance due to an alternating gate operation is superior to the pressure of the direct blow only in the zone Ω_1 limited by the set of curves z_{2i+1}. The comparative study of the pressures of fractional closure and of alternating operation will need to extend to this zone only.

The approximate value of the pressure of fractional closure, corresponding to (91) will be

$$\zeta^2_{2n+1} = 1 + 2\frac{\rho}{\theta} \qquad (91)$$

The upper limit of the pressure of resonance being

$$Z_1^2 = \frac{2\theta^2}{\theta^2 + (\theta - 1)^2} \qquad (77)$$

the condition $Z_1^2 \gtrless \zeta^2_{2n+1}$ leads to

$$\rho \lesseqgtr \frac{1}{2}\,\frac{2\theta^2 - \theta}{2\theta^2 - 2\theta + 1} \qquad (102)$$

equation of a locus, the absissas ρ of which are equal 0.5 for $\theta = 1$ and $\theta = \infty$ and pass through a maximum $\rho = 0.6$ at about $\theta = 1.7$.

In reality, the zone where the limiting pressure of resonance due to an alternating gate operation is greater than the pressure of fractional closure, is greatly reduced, due to the fact already that formula (91) is erroneous for $\rho : \theta$ smaller than 0,5.

The locus R, along which we have effectively

$$\zeta^2_{2n+1} = Z_1^2$$

therefore, must be determined point by point in plotting, on the diagram of fractional closure, the horizontals $Z_1^2 =$ const., of the diagram of resonance due to an alternating operation.

This plotting was done on fig. 62; it was limited to small values of ρ and θ on the theory that this study is essentially interesting in the case of high and very high head conduits situated to the left of the locus R, which approaches the θ axis indefinitely for increasing values of θ.

For such conduits, the maximum pressure of resonance, therefore, is that produced by the alternating gate operation and not that of the fractional closure.

To complete this synopsis of comparison, we have also shown the locus r of fig. (53), to the left of which are located the conduits for which the pressure of resonance due to an alternating gate operation is also greater than the pressure of sudden closure.

Remark

In all the researches exposed in § 's 24, 25 and 26, on the subject of the maximum pressure of fractional closure, we have considered the pressure at the instant of complete closure only, in this manner excluding, a priori, the possibility of an intermediate maximum in the last phase, i. e., in always giving the plot of the pressure the form of the diagrams fig's. 59 bis and 60 bis· This, in fact, is correct within the synoptic field and can be proven analytically. This proof, however, is omitted.

§ 27. — Fractional rhythmic opening.

The phenomena of resonance due to a fractional rhythmic opening present only a mediocre interest from a technical point of view, inasmuch as they involve no great superpressures nor great subpressures. The transformation of the potential energy of the conduit into the kinetic energy which characterises the opening operation, follows, in this case, a rhythmic law with intervals of stoppage at the ends of which the increase of energy of the water column produces a positive counterblow. This counterblow must have a diminishing tendency as the increase of the efflux opening more and more facilitates the flow of the liquid column.

Consequently, we can expect that the phenomenon may be of some interest during the first phases of the gate operation for conduits storing a large quantity of potential energy, i.e., for conduits of high and very high heads (small values of ρ).

We shall discuss the case of fractional rhytmic opening for placing the conduit in service, that is, the fractional opening beginning with the state of rest of the liquid column. This is the most characteristic case and the most important. The reader is reminded of the formulas and graphic constructions of § 17.

First of all, regarding the form of the circular diagram in the case of an opening for placing in service (see also fig. 34, 35, 36) it is easy to see that, for the case of fractional, rhythmic opening, it will be modified in te following manner:

(*A*)	The centers of odd indices	C_1^*	C_3^*	$C_5^* \ldots$
	with abscissae	0	$+\varepsilon_*$	$+2\varepsilon_* \ldots$
	and ordinates	$-\varepsilon_*$	$-2\varepsilon_*$	$-3\varepsilon_* \ldots$

are here yet alined on a straight line at 45° passing above the origin 0, and consequently, all circles γ^*_i with odd indices pass trough the same point M^* of the bissectrix of the axes;

(*B*)	The centers of even indices	C_2^*	C_4^*	$C_6^* \ldots$
	of abscissae	$-\varepsilon_*$	$-2\varepsilon_*$	$-3\varepsilon_* \ldots$
	and ordinates	$+\varepsilon_*$	$+2\varepsilon_*$	$+3\varepsilon_* \ldots$

are situated on the bissectrix of the exterior angle of the axes; all circles γ^*_i of even indices, therefore, pass through the same point K_0 of the bissectrix of the axes, at a distance from the origin O of $OK_0 = \sqrt{2}$ (the coordinates of K_0 are unity).

Let us now examine the diagram of fig. 63, which is constructed for a rather small value of $\varepsilon_* = \frac{\rho_*}{\theta}$ say 0.15 (*); it is evident that on the hypothesis of fractional opening, while the ζ_i of odd indices remain below 1, the first ζ_i of even indices may rise above 1.

In fact, in figure 63 we really have $\zeta_2 > 1, \zeta_4 > 1$ while $\zeta_6 \backsim 1$ and the successive ζ_{2i}'s, determined by the circles γ_{2i}^*, become < 1 and diminish progressively.

This decrease is a great deal more accentuated for larger values of the parameter ε_*, as shown in fig. 64, where $\varepsilon_* = 0.4$; it is shown that ζ_4 is already about equal to 1, while ζ_6 is notably < 1.

It is also easy to see that for $\varepsilon_* \gtrless 1$ it results from the position of C_2^* that already ζ_2 is $\lesseqgtr 1$, and consequentty, the problem is of no interest.

We, therefore, will turn our attention to the conduits and to the gate operation characterized by small values of ε_* and will expecially examine the series of the pressures $\zeta_4^2, \zeta_6^2 \ldots$ etc.

In fact, the study of the value of the first counterblow ζ_2^2 has been fully discussed in § 21 (Note IV) because it is nothing else than the first counterblow (**) after the stoppage of the gate succeeding a sudden opening; it is evident that the continuation of the opening after the counter blow has taken place, can not have any influence upon its value.

The system of equations (56), § 17, which determines the interlocked series of total rhythme of the continuos opening operation for placing in service, evidently becomes, in the case of a fractional opening

$$
\begin{aligned}
\zeta_1^2 - 1 &= -2\,\varepsilon_*\,\zeta_1 \\
\zeta_1^2 + \zeta_2^2 - 2 &= 2\,\varepsilon_*\,(\ \zeta_1 - \ \zeta_2) \\
\zeta_2^2 + \zeta_3^2 - 2 &= 2\,\varepsilon_*\,(\ \zeta_2 - 2\zeta_3) \\
\zeta_3^2 + \zeta_4^2 - 2 &= 2\,\varepsilon_*\,(2\,\zeta_3 - 2\zeta_4) \\
\zeta_4^2 + \zeta_5^2 - 2 &= 2\,\varepsilon_*\,(2\,\zeta_4 - 3\zeta_5) \\
\zeta_5^2 + \zeta_6^2 - 2 &= 2\,\varepsilon_*\,(3\,\zeta_5 - 3\zeta_6) \\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
\qquad (103)
$$

in which equations, as will be remembered, $\varepsilon_* = \frac{\rho_*}{\theta_*}$, (see § 17), ρ_* and θ_* being the values of the characteristics for the regimen attained by the opening operation.

By means of (103) we should now find the conditions of maxima of $\zeta_4, \zeta_6 \ldots$ etc., but an analytical research is already excessively complicated for the sole value of ζ_4, while the same result can be obtained more easily by numerical trials made systematically for small values of $\frac{\rho}{\theta}$ comprised between 0.05 and 0.75.

(*) Which would be realized by a conduit of $y_0 = 200$ to 300^m $L = 700$ to 1000^m, whic would need a time to from 4 to 6 seconds to produce a velocity of regime v_0 of from 2 to 3^m by a continous opening operation.

(**) In order to understand the subject more clearly, the reader will recollect that in § 22 we have designated this counterblow by ζ_1^2 as being the first pressure of total rhythme after the stoppage of the gate, while in the present case this pressure has the index 2, being the second pressure of total rhythme of the series of pressure of fractional opening.

1st trial $\varepsilon_* = 0,05$

that is, for	$\rho_* = 0,10$	$0,15$	$0,20$	etc.
	$\theta_* = 2,00$	$3,00$	$4,00$	»
last term	ζ_4	ζ_6	ζ_8	»

By means of the system (103) we obtain the following values of the pressures of the total rhytm:

ζ_1^2	ζ_2^2	ζ_3^2	ζ_4^2	ζ_5^2	ζ_6^2	ζ_7^2	ζ_8^2
0,904	—	0,835	—	0,809	—	0,818	—
—	1,086	—	1,134	—	1,141	—	1,121

which shows that the maximum pressure occurs for $\rho_* \gtrless 0,15$ and $\theta_* \gtrless 3$.

Consider now the pressure $\zeta_4^2 = 1,134$; it is clear that for the conduits ($\rho_* = 0,10$, $\theta = 2$) the fractional gate opening results in a superpressure equal to 2/3 of that of sudden closure.

2dn trial $\varepsilon_* = 0,10$

that is, for	$\rho_* = 0,2$	$0,3$	$0,4$	etc.
	$\theta_* = 2$	3	4	»
last term	ζ_4	ζ_6	ζ_8	»

We obtain with the help of (103)

ζ_1^2	ζ_2^2	ζ_3^2	ζ_4^2	ζ_5^2	ζ_6^2
0,819	—	0,726	—	0,740	—
—	1,147	—	1,179	—	1,136

which clearly shows that for increasing values of ε_*, the maximum positive counterblow occurs at an instant approaching the beginning of the operation. In fact, it is not anymore the pressure of index 6, but that of index 4 which has a maximum value; the maximum superpressure for $\rho_* = ,2$ and $\theta = 2$ is about equal the half of the superpressure of the sudden closure.

3rd trial $\varepsilon_* = ,15$

that is, for	$\rho_* = 0,3$	$0,45$	$0,6$	etc.
	$\theta_* = 2$	3	4	»
last term	ζ_4	ζ_6	ζ_8	»

We obtain from (103)

ζ_1^2	ζ_2^2	ζ_3^2	ζ_4^2	ζ_5^2	ζ_6^2
0,741		0,653	—	0,712	—
—	1,190	—	1,179	—	1,102

The pressure with index 2 becomes the maximum pressure of the counter blow, while that of index 6 continues to diminish. We can observe how, by the decrease of potential energy, the phenomenon of resonance due to a fractional rhythmic opening gradually loses intensity and extension.

4th, 5th, and 6th trials.

	ζ_1^2	ζ_2^2	ζ_3^2	ζ_4^2	ζ_5^2	ζ_6^2
$\varepsilon_* = 0{,}25$	0,610	1,225	0,572	1,124	0,600	1,032
$\varepsilon_* = 0{,}50$	0,382	1,160	0,501	0,958		
$\varepsilon_* = 0{,}75$	0,250	1,00	0,642	etc.		

which results show the rapid decrease of the intensity of the phenomenon with increasing values of ρ_*.

Beyond $\rho_* = 0{,}75$ the phenomenon loses all interest, inasmuch as for such conditions all pressures are smaller than unity.

INTRODUCTION

In December 1902, Mr. Lorenzo Allievi, C. E. published his monograph *Teoria Generale del moto perturbato dell'acqua nei tubi in pressione* (General Theory of the variable motion of water in pressure conduits) in the « *Annali* della Società degli Ingegneri ed Architetti Italiani ». A French translation, by the author himself, appeared in 1904 in the « *Revue de Mecanique* ».

To the writers knowledge, no English translation of this paper was ever published; however, in recent years, one of Allievi's formulas applicable to a specific case was repeatedly quoted in the American technical press as representing « Allievi's solution » of the waterhammer problem. The improper application of this formula caused a great deal of confusion and this, in turn resulted in frequent and unjust criticism of « Allievi's solution », until, in the course of his discussion of Mr. N. R. Gibson's excellent, paper on « Pressures in Penstocks » (Transactions A. S. C. E. Vol. LXXXIII), the writer has given a brief summary, and indicated the method of the practical use, of the formulas derived by Allievi in his 1902 paper.

So far as the writer is aware, Mr. N. R. Gibson's paper was the first one published in the English language which gave a true account of the time history of the pressure wave generated in a penstock during the phenomena of waterhammer; as stated in his discussion, the results obtained by Mr. Gibson are identical with those found by Allievi's formulas.

This remarkable study (Allievi 1902) differs from all monographs previously published on the important question of waterhammer bothas to the originality of the method employed and as to the importance and novelty of the obtained results. Not wishing to use the paths already broken by his predecessors, and aiming primarily to come as close as possible to the solution of the phenomenon, Allievi starts by systematically and knowingly ignoring all that was accomplished before him; he takes up the problem at its origin and presents it as his remarkable intuitional qualities make him foresee that it must be.

However, as judiciously stated by the Author at a lecture given on this subject in 1911 to a group of engineers at Geneva, the paper of 1902-1904 does not constitute a « Theory of Waterhammer », it is only its « mathematical tool ». In the Notes, the translation of which is here undertaken, and which were published in the « Atti del Collegio degli Ingegneri ed Architetti », Milano, 1913, Allievi makes a wonderful use of this tool forged by himself, and presents to his readers this « General Theory », the fruit of his recent researches.

By eliminating all arbitrary assumptions in the derivation of the fundamental equations of the waterhammer, Allievi succeeded in giving a mathematical translation of the phenomenon, which is the exact expression of its physical features; in this lies the originality of the author's first work (1902-1904).

This method of facing the problem, together with the interpretation of the hydrodynamic function played by the reservoir of constant pressure situated at the upstream end of the pipe, which function consists in reflecting toward the gate (with opposite signs) the waves of surpression and depression carried there by the conduit, has given, so to speak, the key of all the waterhammer phenomena.

Allievi's monograph of 1902, of which a brief summary will be given following this introduction, permits already to determine analitically the character of a waterhammer due to any kind of operation of a gate regulating the efflux of the conduit; but as already stated above it is substantially only the mathematical tool of a theory of waterhammer; it does not yet contain a systematic and synthetic study of the general laws of the phenomenon.

As a matter of fact, in this first monograph, only those elements which mechanically characterize the conduit, such as the diameter, the thickness of the pipe-shell and the moduli of elasticity of the metal and liquid, are absorbed in the expression of the speed ot propagation a of the waterhammer; while the elements which characterize the functioning of the conduit, that is the pressure height y_0 and the velocity of permanent motion v_0 are still in evidence, which makes impossible all systematic generalization of the laws governing the pressures during the perturbed flow.

There arose then a new difficulty, seemingly considerable, a ditch, which the author has crossed wih great elegance in his new researches; the publication of which in the English language is here undertaken.

With the help of a quite elementary transformation of his fundamental formulas, Allievi demonstrates that, no matter what be the gate operation, the relative values of the pressure in the disturbed regimen (i. e. the ratio of the pressure to its initial value) depend, first, on the method of operation, in other words, on the series of the degrees of openings of the gate, and, second, on a single parameter, equal of one half of the square root of the ratio of the kinetic energy and the potential energy contained, during normal flow, in a unit length of the conduit.

This parameter, which Allievi designates by ρ and which is equal to

$$\rho = \frac{a\, v_0}{2\, g\, y_0},$$

absorbs all constructive and funtional elements of the conduit; it is therefore justified that the author calls it the « characteristic » of all conduits in service.

There is, in this simple remark, an extremely fertile and powerful instrument for the generalization of the phenomenon, as it is evident that an infinite number of conduits, the diameter, thickness, elastic constants, normal pressure and velocity of which result in the same value of the characteristic ρ will obey the same system of laws in regard to the phenomena of the waterhammer.

Hovewer, the conduits so corresponding would yet differ from each other, from the point of view of these phenomena, by their lenghts L. Allievi eliminates this difficulty by adopting, as the unit of time, the phase, or half-rytm $\mu = \frac{2\,L}{a}$, of the pressure oscillations.

By help of these transformations, the laws of pressure during the variable motion are so expressed as to be functions of only two parameters; the characteristic ρ, and the time θ of closing (or opening) of the gate measured by the new unit of time just defined. This transformation also permits of the plotting of graphs where, in a cartesian system of axes θ and ρ, all laws of the waterhammer, for all possible conduits and all speeds of gate operation imaginable can be represented by curves, the ensemble of which constitutes the diagrams of the different phenomens and different categories of the conduits.

These diagrams, which the author calls « cartesian synopsis » constitute the most impressive results of these new theories; the engineer will find that they give as simple means as imaginable for solving in a few second and with the greatest ease, all the problems relating to this class of phenomena.

Another instrument of graphic analysis, equally fertile and one which the reader will much appreciate, consists in what the author calls the « circular diagrams ».

Adopting as the unknown the square root of the relative value of the pressure (in other words the relative value of the velocity of efflux through the gate) Allievi demonstrates that it is easy to obtain a series of values of the pressure during the perturbed flow by drawing a series of circles.

The extraordinary fertility of this method appears among others in the fact that the drawing of a single circle is sufficient to determine.

1st — The maximum and average pressures produced by a closing operation.

2nd — The maximum and average pressures produced by an opening operation.

3rd — The limiting pressures maximum and minimum, of the resonance.

Part of this introduction, upon the Author's request, was taken from the Preface of the translation by R. Neeser, Professor of the University of Lausanne.

The following abstract of Allievi's 1902 paper is also by Mr. Neeser, and is translated from the Frenh original.

ABSTRACT OF MR. ALLIEVI'S PAPER (1902):

GENERAL THEORY OF THE VARIABLE FLOW OF WATER IN PRESSURE CONDUITS

In the following short abstract of part of Allievi's 1902 monograph, the genesis of the fundamental equations which constitute the starting point of the « Theory of Waterhammer » will be demonstrated. This will serve to acquaint the reader with the formulas which are referred to as assumedly known in § 1 of Note 1.

1. *Differential equations and fundamental equations of the variable flow of water.*

Consider a horizontal, indefinitely long cylindrical conduit of constant diameter and wall thickness, having a gate at one end which is able to close or open the discharge orifice. Let us also assume that the influence of the friction of the water against the perimeter can be neglected; this hypothesis is perfectly permissible in all cases where the loss in head due to friction is negligible compared to the pressure intensities accompanying the phenomena of the variable motion.

Further, let

- r, D, e, be the geometrical elements of the conduit: radius, diameter, thickness;
- E, the modulus of elasticity of the conduit wall;
- ε, ω, » » » » and the specific gravity of the flowing liquid;
- v_0, p_0, the velocity and pressure of uniform flow (regimen), before the perturbance;
- v, p, the velocity and pressure at any instant and at any section;
- y, y_0, the pressures, expressed in height of water column, corresponding respectively to the variable regimen and to the initial uniform regimen.

Let x be the abcissa of any section, measured along the pipe from the orifice toward the reservoir, (in the direction opposite to that of the flow); then the general equation of the variable flow is;

$$\frac{\delta p}{dx} = \frac{\omega}{g}\left(X - \frac{\delta^2 x}{\delta t^2}\right),$$

or, considering the horizontal position of the x axis and the assumed direction of x,

$$\frac{\delta p}{\delta x} = \frac{\omega \delta^2 x}{g \delta t^2}$$

which equation can also be written

$$\frac{\delta p}{\delta x} = \frac{\omega}{g}\left(\frac{\delta v}{\delta t} - v\frac{\delta v}{\delta x}\right), \qquad \text{(I)}$$

because v is a function of x and t.

The equation of continuity will furnish a new relation; it is sufficient to express that the difference of the volumes of water flowing, during the time dt, across two sections of the conduit a distance dx apart, is equal to the volume (of water) stored, during that time, in the element dx of the conduit; but the variation of the volume of this element is made up of.

$$\frac{\pi r^2}{E} \cdot \frac{D}{e} \cdot \frac{\delta p}{\delta t} \delta t \,.\, dx,$$

resulting from the elasticity of the conduit wall, and of

$$\pi r^2 \frac{dx}{\varepsilon}\frac{\delta p}{\delta t} dt,$$

due to the compressibility of the liquid.

Neglecting the terms in $\frac{1}{E^2}$ and the differentials higher than the first order, one thus obtains:

$$\frac{\delta v}{\delta x} = \left(\frac{1}{\varepsilon} + \frac{1}{E}\frac{D}{e}\right)\frac{\delta p}{\delta t} \qquad \text{(II)}$$

These equations (I) and (II) are the fundamental differential equations of the variable flow; they can be reduced to a simpler form by neglecting in equation (1), the term in $\frac{dv}{dx}$; this simplification is permissible, because the phenomena of waterhammer occur almost instantaneously, so that the variation of v, with respect to the abcissa x is certainly negligible compared to the variation of the velocity with respct to the time.

Finally, putting

$$\frac{\omega}{g}\left(\frac{1}{\varepsilon} + \frac{1}{E}\frac{D}{e}\right) = \frac{1}{a^2} \qquad \text{(III)}$$

in which equation a, as it is easy to prove, has the dimensions of a velocity, equations (I) and (II) become:

$$\left.\begin{aligned} \frac{\delta v}{\delta t} &= g\frac{\delta y}{\delta x} \\ \frac{\delta v}{\delta x} &= \frac{g}{a^2}\frac{dy}{dt} \end{aligned}\right\} \quad \text{(IV).}$$

The general integrals of equations (4) as can be readily verified, can be written

$$\left.\begin{aligned} y &= y_0 + \mathrm{F}\left(t - \frac{x}{a}\right) + f\left(t + \frac{x}{a}\right) \\ v &= v_0 - \frac{g}{a}\left[\mathrm{F}\left(t - \frac{x}{a}\right) - f\left(t + \frac{x}{a}\right)\right] \end{aligned}\right\} \quad \text{(V).}$$

Equations (IV) and (V) indicate that the phenomenon of waterhammer is characterized by two systems of co-existent variable pressures which are propagated along the conduit in opposite directions and with the constant velocity a. In fact, putting $x = + at +$ constant, or $x = - at +$ constant, makes the function F or the function f constant, and also the values of y, and of v depending on both of these systems of pressures.

We will designate by « direct blow » the one dependent on the first of these systems of pressures, and which, due to a variation of the section of the orifice, is propagated from the orifice toward the reservoir in the direction of positive x's and we will designate by « counter blow » the waterhammer which, due to the reaction of the reservoir, is propagated from the reservoir tovard the orifice in the direction of negative x's.

2. *The Waterhammer of the direct blow.*

If the conduit is indefinitely long, in other words if the reservoir is indefinitely distant, there cannot occur, at *no* point of the conduit, a reflected wave, and the conditions of the direct blow will be always fulfilled.

If, however, the conduit has a finite length L, the waterhammer of the direct blow, at any section of abcissa x, will have only a limited duration of the time $\frac{2\,\mathrm{L} - x}{a}$.

In the case of the direct blow equation (V) reduces to

$$\left.\begin{aligned} y &= y_0 + \mathrm{F}\left(t - \frac{x}{a}\right) \\ v &= v_0 - \frac{g}{a}\mathrm{F}\left(t - \frac{x}{a}\right) \end{aligned}\right\} \quad \text{(VI)}$$

3. *Waterhammer of the counter blow.*

In a conduit of the finite length L, supposed to be horizontal and fed by a reservoir of constant pressure y_0, at any section of abcissa x and beginning with the time

$$t = \frac{2L - x}{a},$$

the equations of the variable motion will become.

$$y = y_0 + F + f$$

$$v = v_0 - \frac{g}{a}(F - f)$$

because, beginning at that instant, there will co-exist at x both the direct and the reflected wave. The functions $F\left(t - \frac{x}{a}\right)$ and $f\left(t + \frac{x}{a}\right)$ are unknown. It is possible, however, to eliminate one of these functions by considering that at the section $x = L$ (at the reservoir), the pressure y must be constant and equal to y_0 whatever the value of t. We then evidently obtain

$$-f\left(t + \frac{L}{a}\right) = F\left(t - \frac{L}{a}\right); \qquad \text{(VII)}$$

or in particular, putting

$$t = t_1 + \frac{x}{a} - \frac{L}{a};$$

where t_1, designates any instant of the phase of the counter blow at section x, provided that

$$t_1 \geq \frac{2L - x}{a},$$

equation (VII) becomes

$$f\left(t_1 + \frac{x}{a}\right) = -F\left(t_1 + \frac{x}{a} - \frac{2L}{a}\right), \qquad \text{(VIII)}$$

which is the characteristic equation of the phase of the counter blow.

The first of the equations (V)

$$y = y_0 + F\left(t - \frac{x}{a}\right) + f\left(t + \frac{x}{a}\right)$$

evidently has a very clear significance; the functions F and f represent travelling pressures which are displaced along the conduit whith a velocity a, the first in the positive and the other in the negative direction of the abcissa. Therefore, the waterhammer, at any section of abcissa x and at any time t, is given by.

$$\Delta y = y - y_0 = F\left(t - \frac{x}{a}\right) + f\left(t + \frac{x}{a}\right).$$

This equation is therefore the mathematical expression of the following fact: the waterhammer at any section and at any time is equal to the sum of the travelling pressures F and f which are displaced in the two directions with the constant velocity a, and without mutually interfering with each other.

The second of the formulas (V),

$$v = v_0 - \frac{g}{a}(F - f)$$

at first glance, does not seem to be susceptible to such a simple interpretation. Allievi, however, remarks, that this second equation can be arrived at by means of elementary methods, if it is admitted, as shown by experience, that the law of propagation of pressures at constant velocity expressed by the first equation, (V), is true. It is sufficient, to this effect, to apply the fundamental principles of dynamics to an element of the liquid column of thickness $dx = a\,dt$. The second relation of (V) can be deduced, without difficulty, from the equation of the motion of this element.

At any instant t, in the horizontal conduit, a fluid element of abcissa x and of thickness dx is subjected to the action of the travelling pressures F and f; the difference of these pressures on te two faces of the element is evidently,

$$dF - df,$$

so that the equation of motion of the element can be written

$$\frac{\omega}{g}\,\pi r^2 dx\,\frac{dv}{dt} = \omega\,(dF - df)\,\pi r^2$$

or putting

$$dx = a \cdot dt$$

$$\frac{a}{g}\,dv = dF - df.$$

Integrating this equation between the limits, $t = 0$, the beginning of the perturbed regime, for which

$$v = v_0 \text{ and } F = f = o,$$

and an instant t, of the phase of the counter blow, we get

$$v = v_0 - \frac{g}{a}(F - f),$$

which is the second equation of the system (V).

Lorenzo ALLIEVI

THEORY

OF

WATER-HAMMER

Translated by Mr. Eugene E. HALMOS
M. Am. Soc. C. E.

NOTES I TO V

(FIGURES)

ROME
TYPOGRAPHY RICCARDO GARRONI
Piazza Mignanelli, 23
1925

THEORY

OF

WATER-HAMMER

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NOTES I TO V

(FIGURES)

ROME
TYPOGRAPHY RICCARDO GARRONI
Piazza Mignanelli, 23
1925

Fig. 1ª

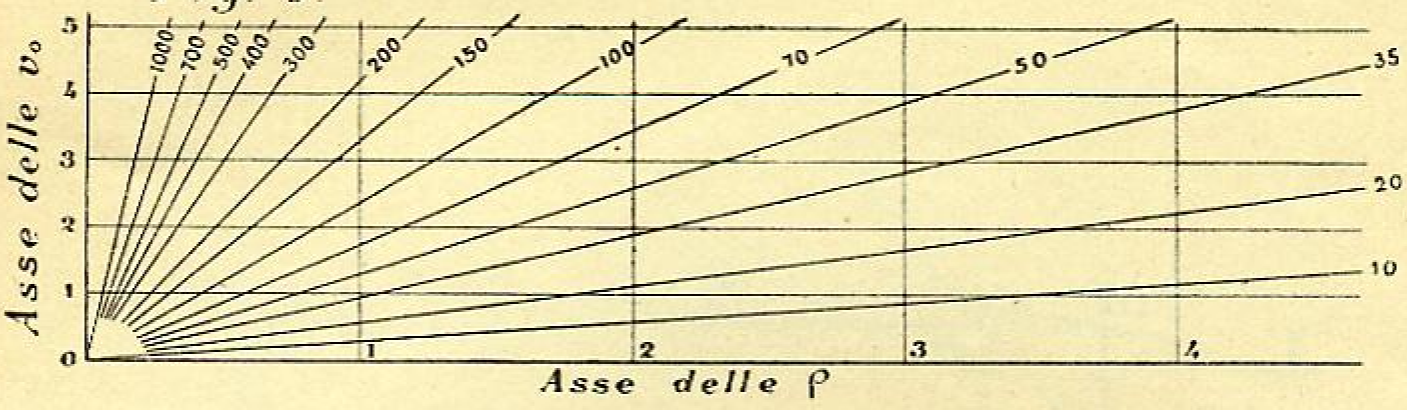

Asse delle v_0
Asse delle ρ
1000
700
500
400
300
200
150
100
70
50
35
20
10
0
1
2
3
4
5

Fig. 2ª

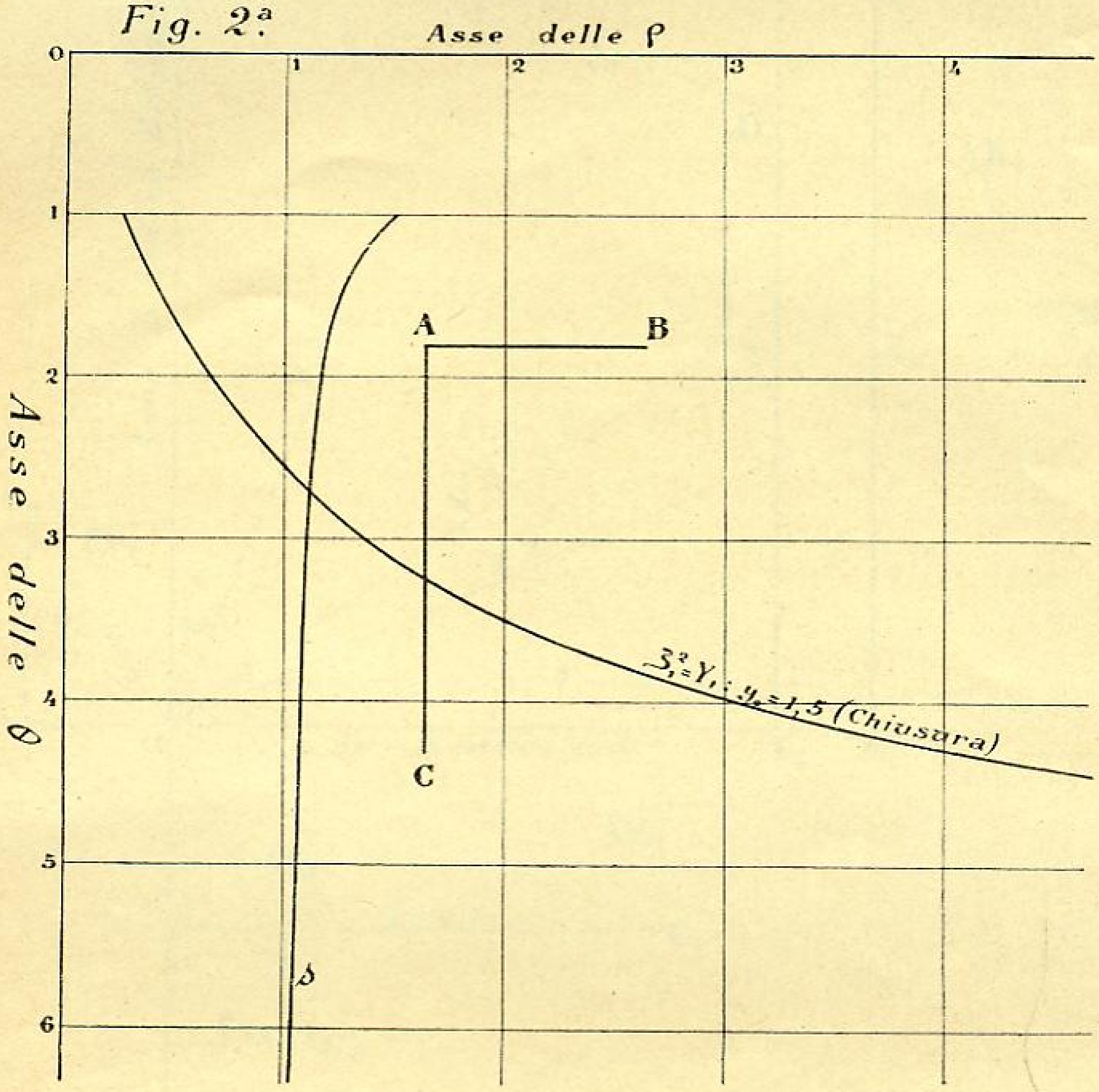
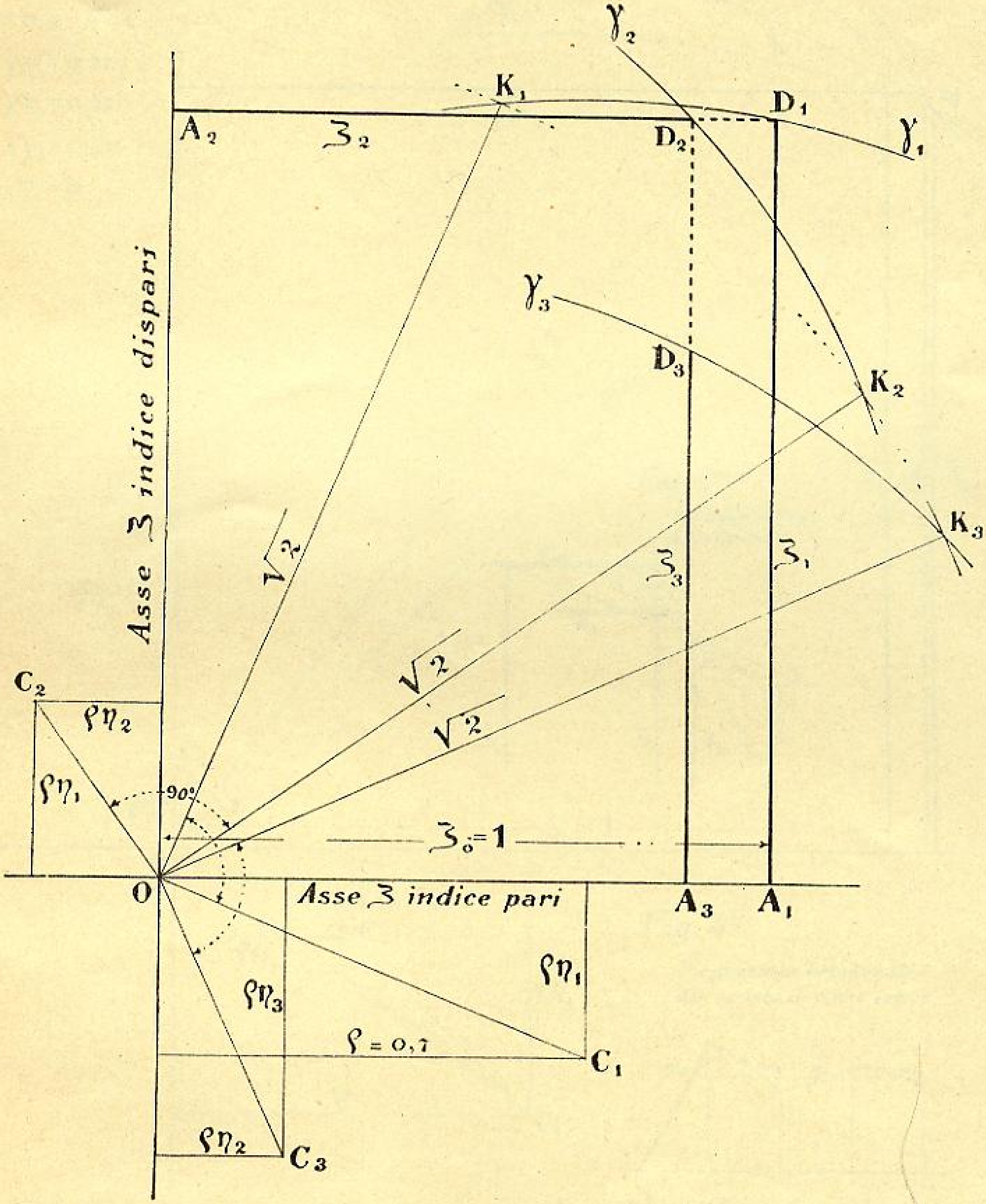

Asse delle ρ
Asse delle θ
0
1
2
3
4
5
6
A
B
C
S
$\mathfrak{Z}_1^2 = Y_1 ; y_1 = 1,5$ (Chiusura)

Fig. 3.ª



γ₂
K₁
D₁
γ₁
A₂
ℨ₂
D₂
γ₃
D₃
K₂
K₃
Asse ℨ indice dispari
√2
√2
√2
ℨ₃
ℨ₁
C₂
ρη₂
ρη₁
90°
ℨ₀=1
O
Asse ℨ indice pari
A₃
A₁
ρη₁
ρη₃
ρ = 0,7
C₁
ρη₂
C₃

Fig. 4ª

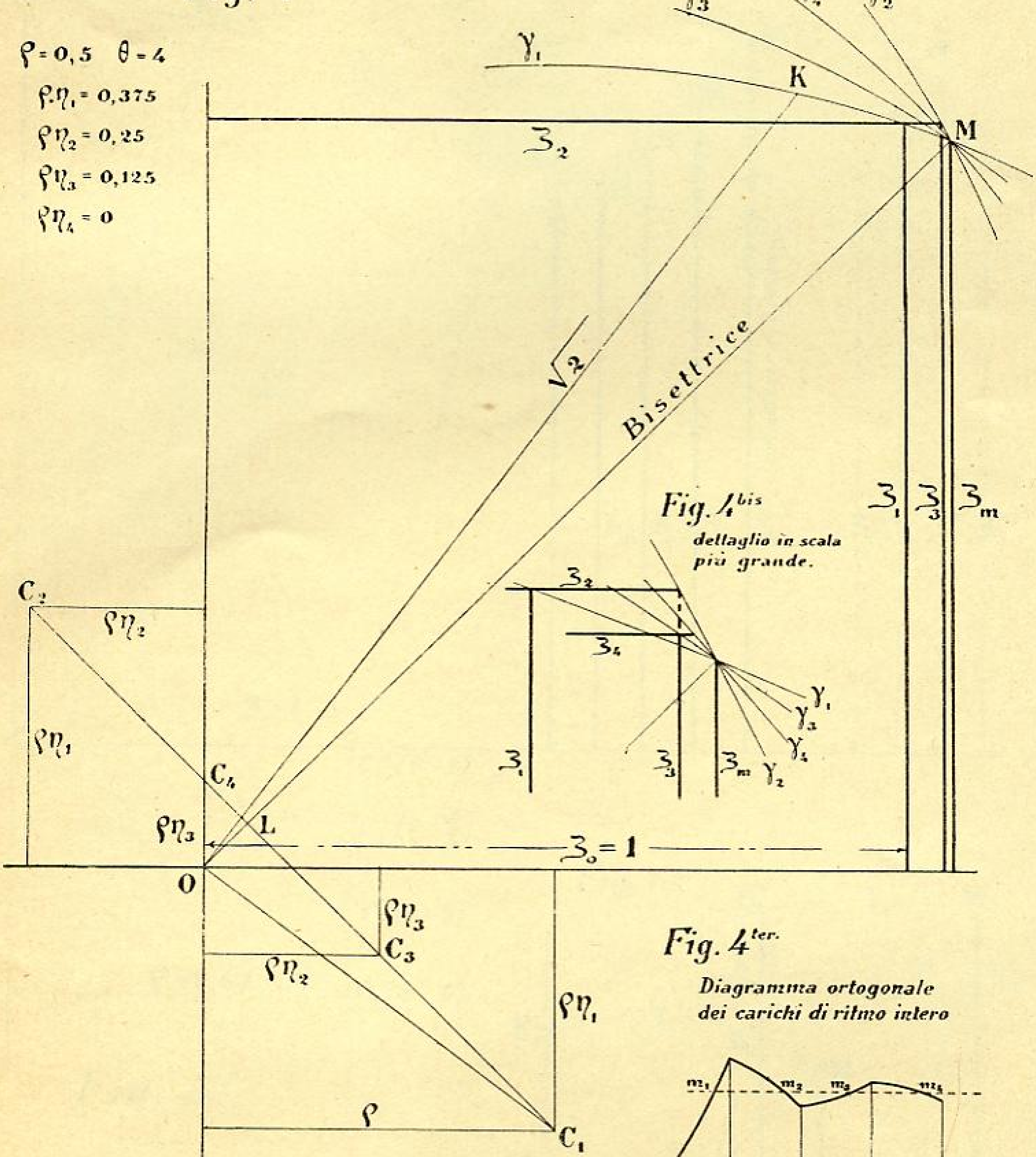
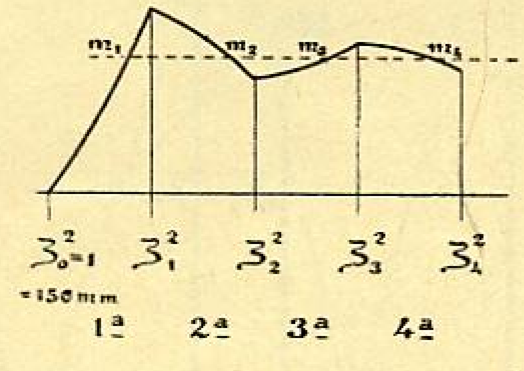


Fig. 4ter.

Diagramma ortogonale dei carichi di ritmo intero



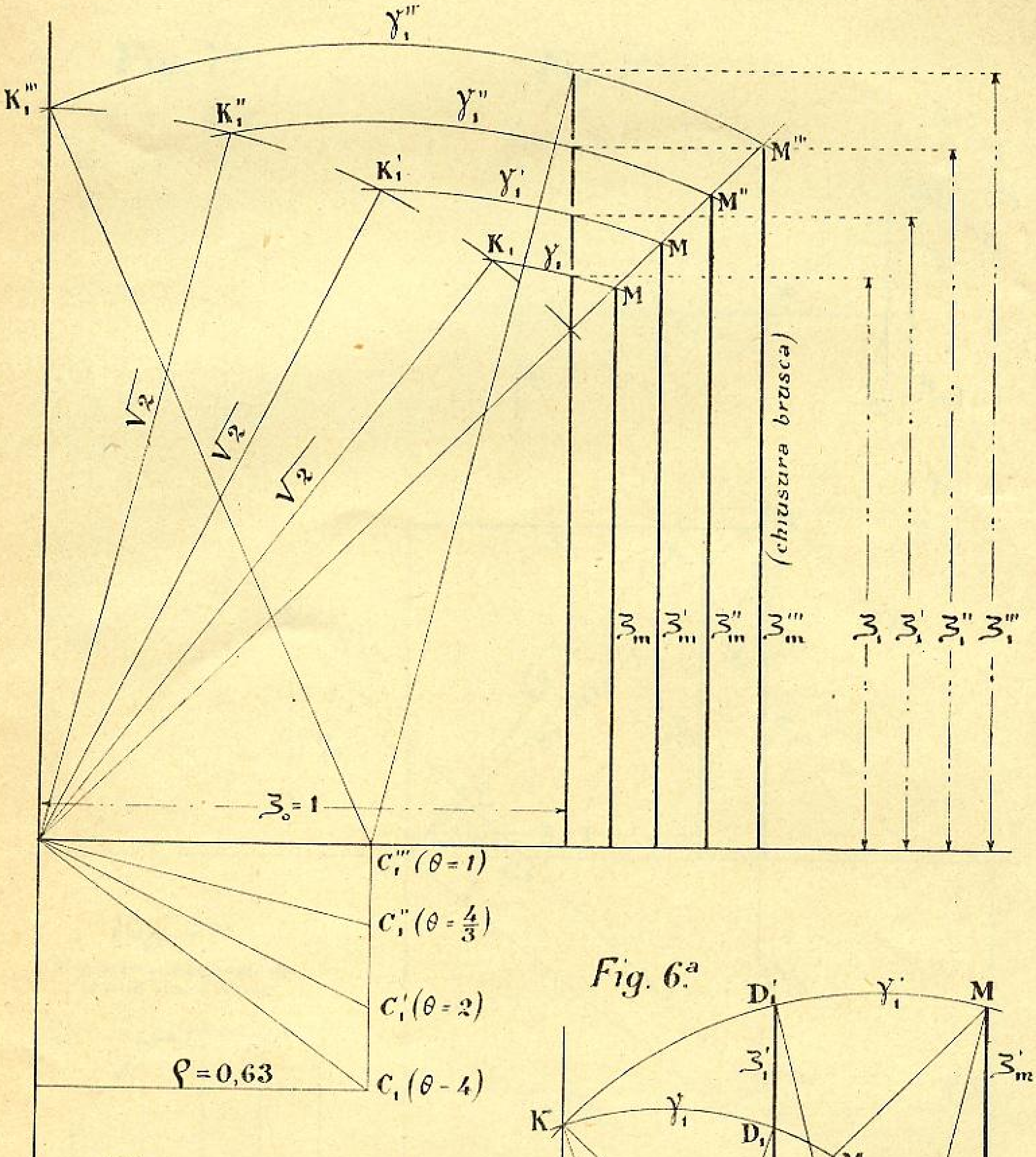

K₁'''
K₁''
K₁'
K₁
γ₁'''
γ₁''
γ₁'
γ₁
M'''
M''
M
M
√2
√2
√2
(chiusura brusca)
z_m z_m' z_m'' z_m'''
z₁ z₁' z₁'' z₁'''
z₀ = 1
C₁''' (θ = 1)
C₁'' (θ = 4/3)
C₁' (θ = 2)
C₁ (θ - 4)
ρ = 0,63

Fig. 5.ª

Fig. 6.ª

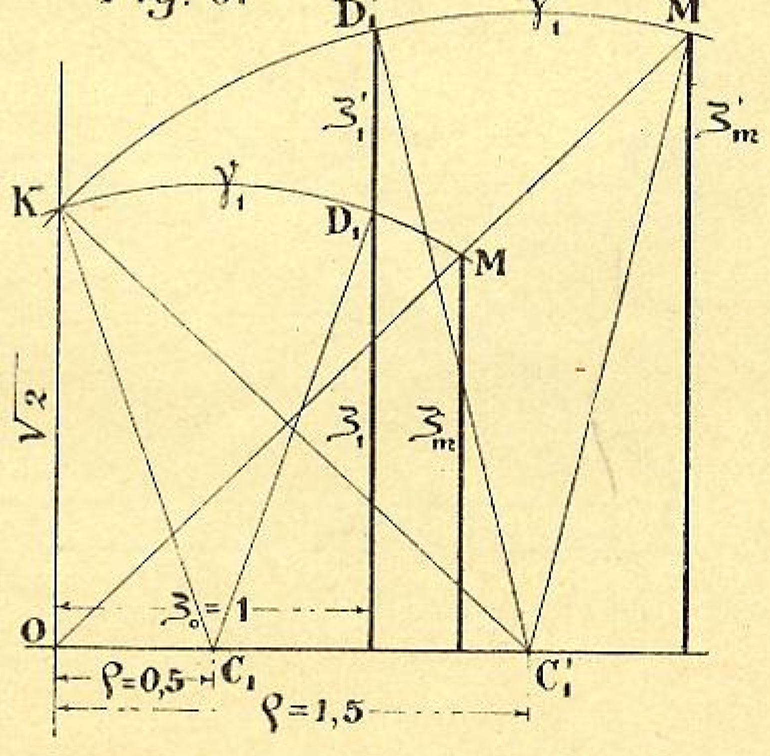

D₁'
γ₁'
M
z₁'
z_m'
K
γ₁
D₁
M
√2
z₁
z_m
z₀ = 1
O
ρ = 0,5
C₁
C₁'
ρ = 1,5

Fig. 7.ª

$\rho = 2 \qquad \theta = 5$

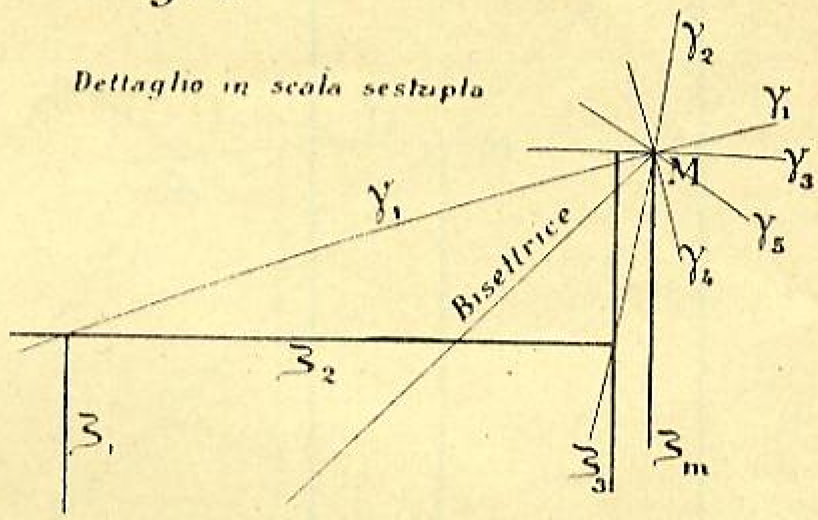
$\rho\eta_1 = 1{,}6 \qquad \rho\eta_2 = 1{,}2$

$\rho\eta_3 = 0{,}8 \qquad \rho\eta_4 = 0{,}4$

$\rho\eta'_5 = 0$

Fig. 7.bis

Dettaglio in scala sestupla



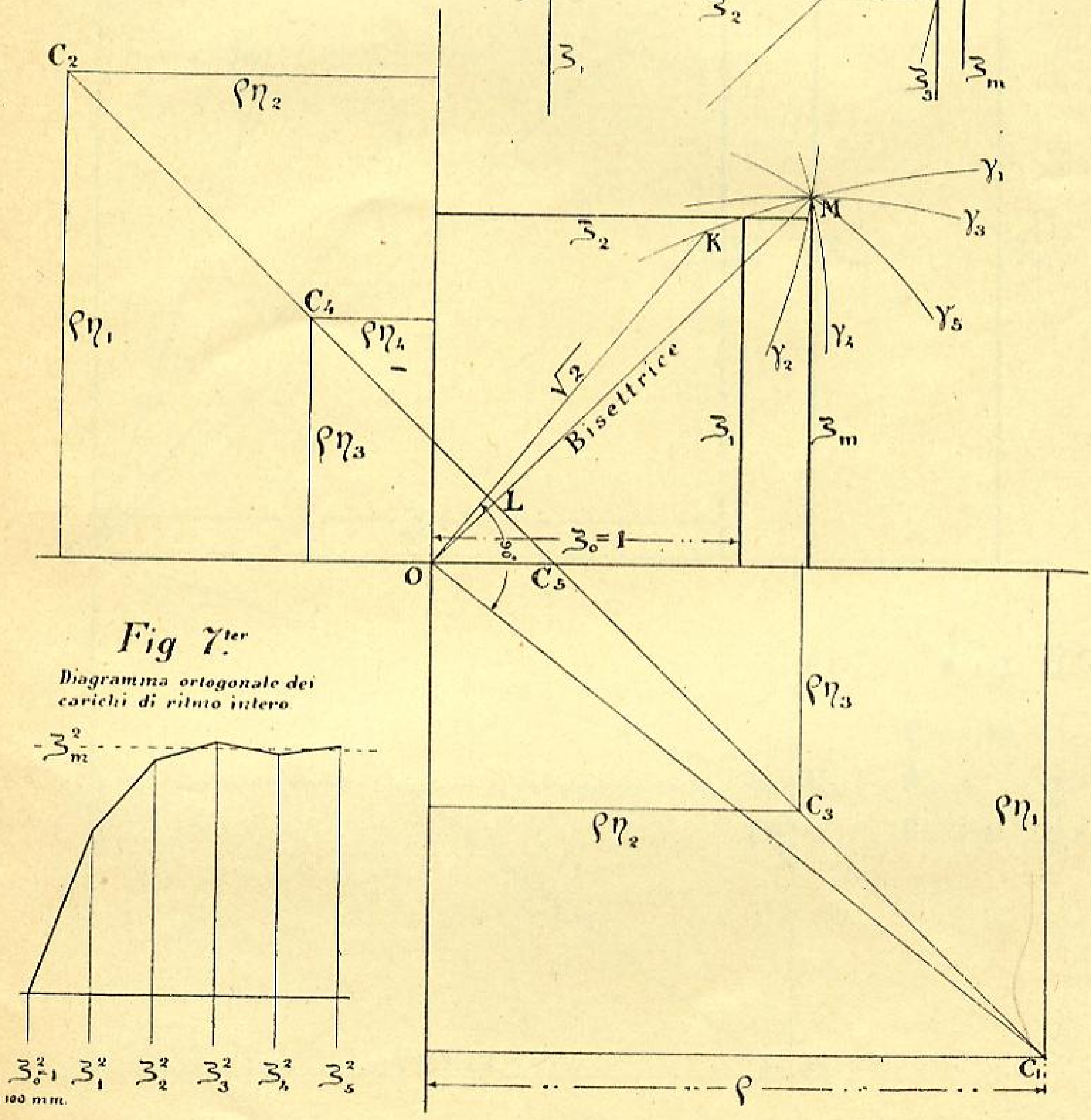


Fig 7.ter

Diagramma ortogonale dei carichi di ritmo intero

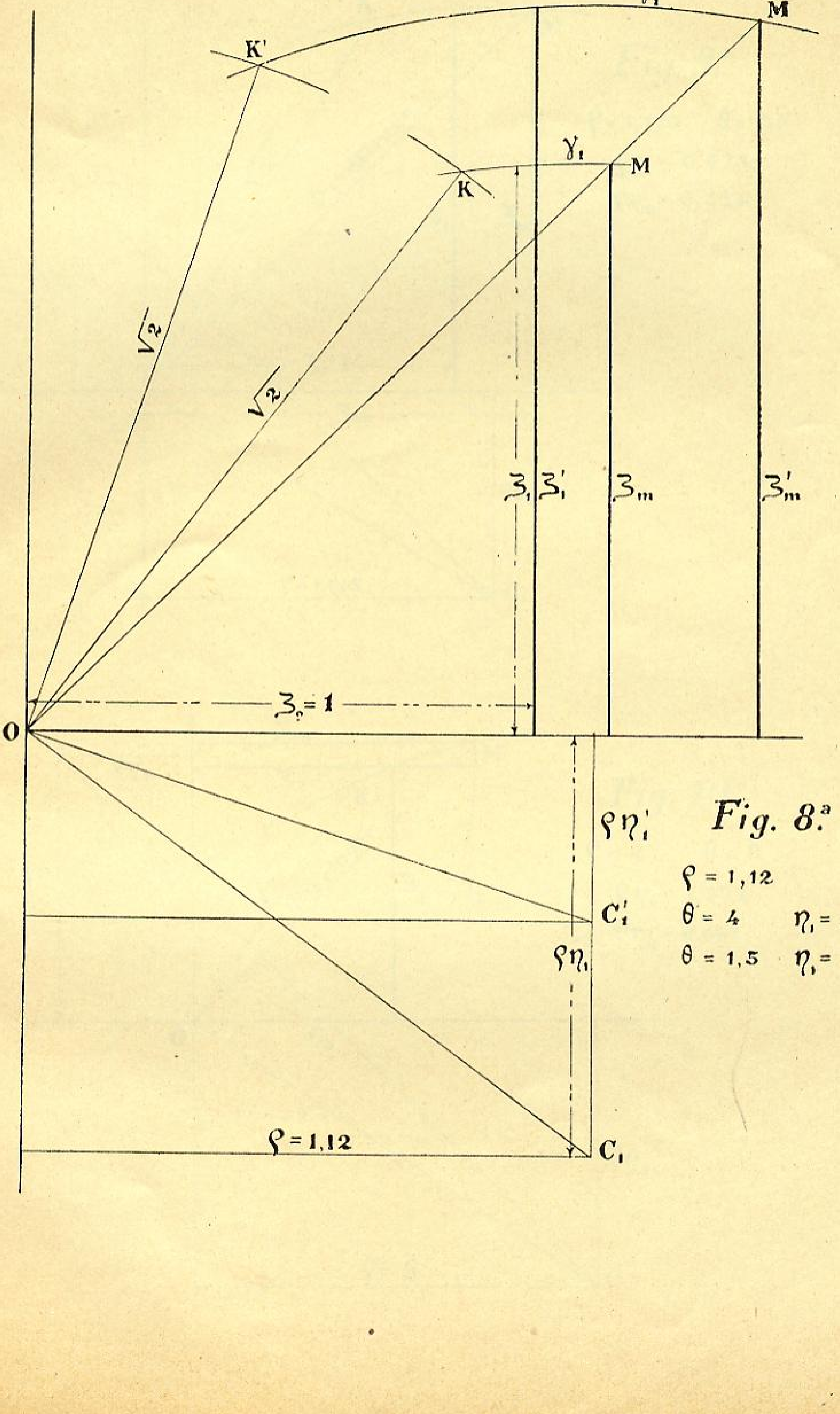


Fig. 8.ª

$\rho = 1,12$

$\theta = 4 \qquad \eta_1 = \frac{3}{4}$

$\theta = 1,5 \qquad \eta_1 = \frac{1}{3}$

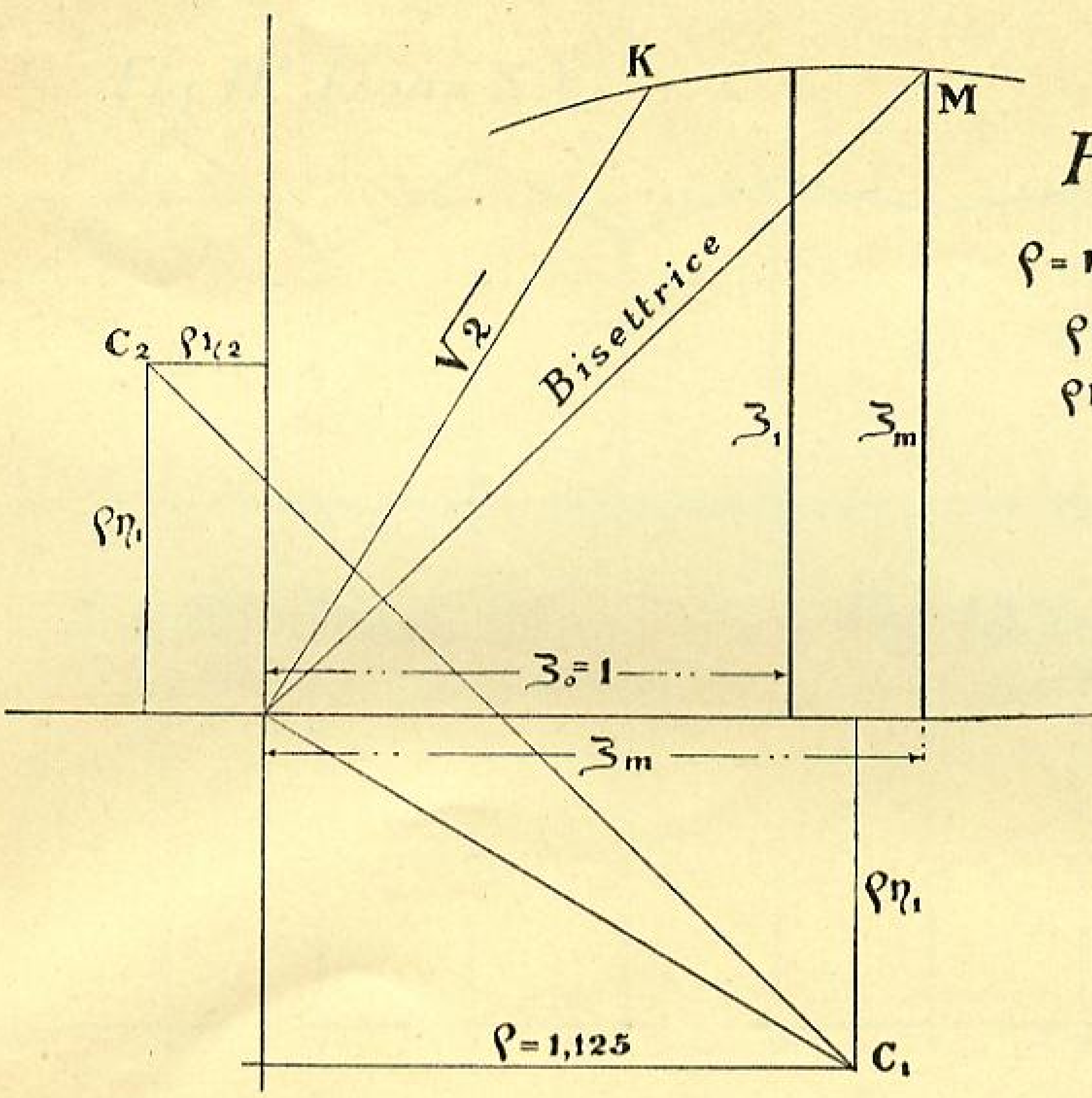


Fig. 9ª

$\rho = 1,125 \quad \theta = 2,5$

$\rho\eta_1 = 0,675$

$\rho\eta_2 = 0,225$

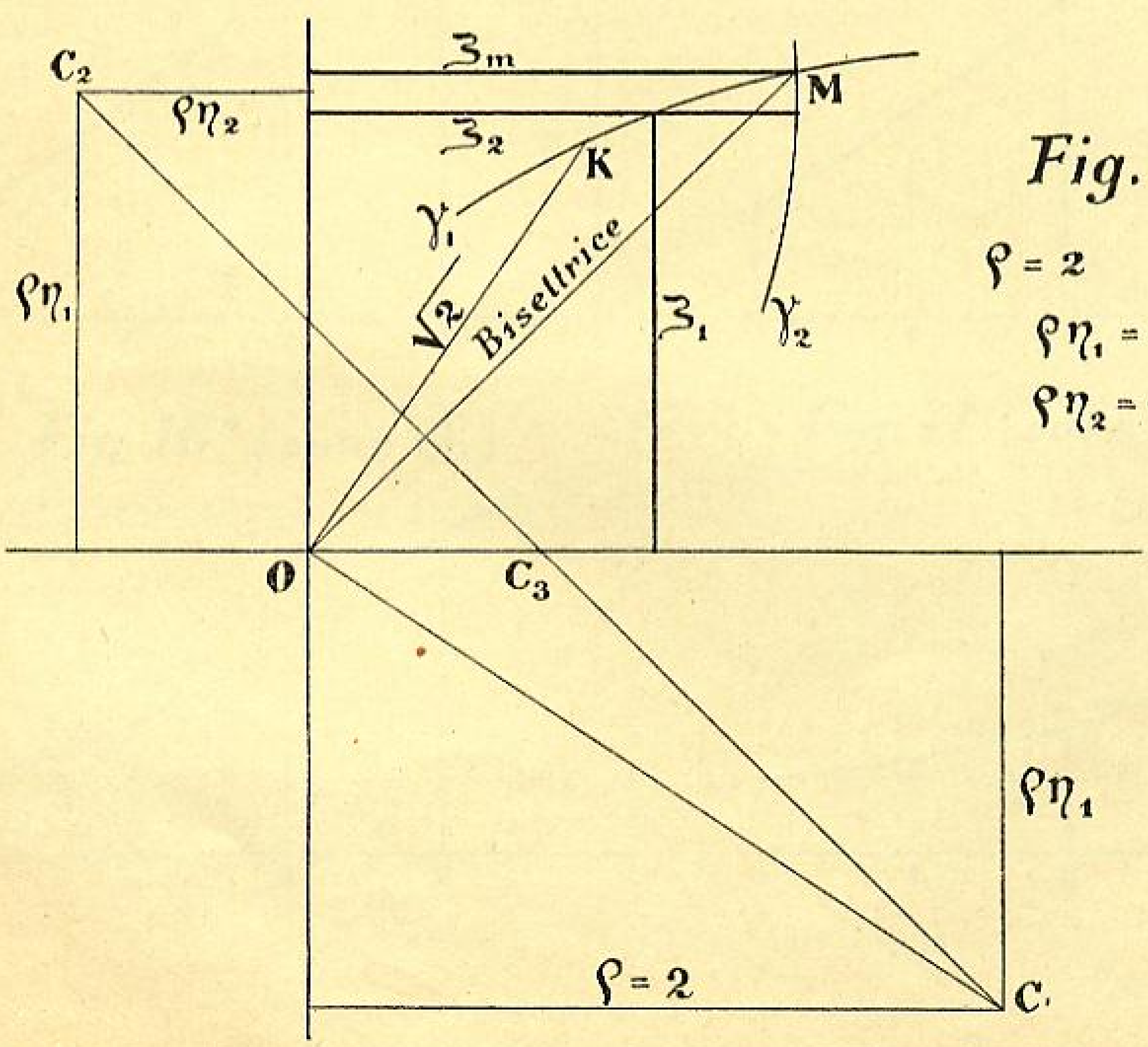


Fig. 10

$\rho = 2 \quad \theta = 3$

$\rho\eta_1 = 1,333$

$\rho\eta_2 = 0,667$

Fig. 11.ª (zona Σ_1)

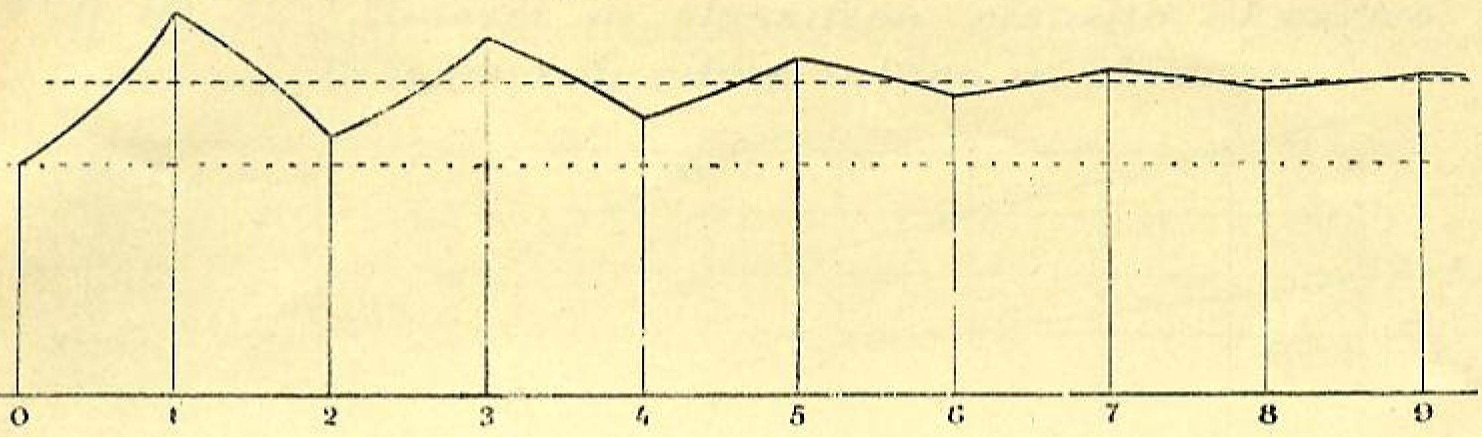


Fig. 12.ª (luogo s_1)

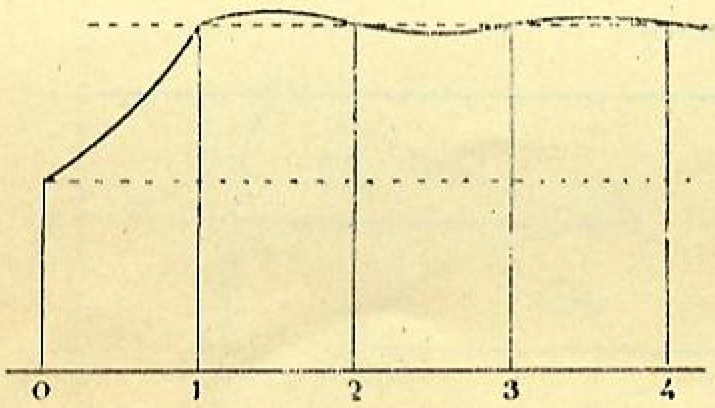


Fig. 13.ª (zona Σ_2)

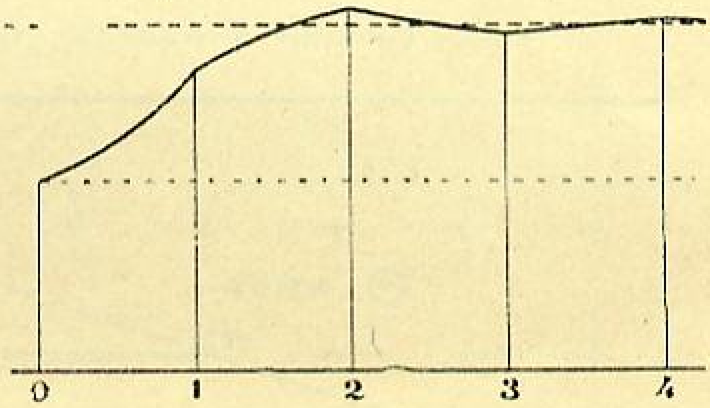


Fig. 14.ª (luogo s_2)

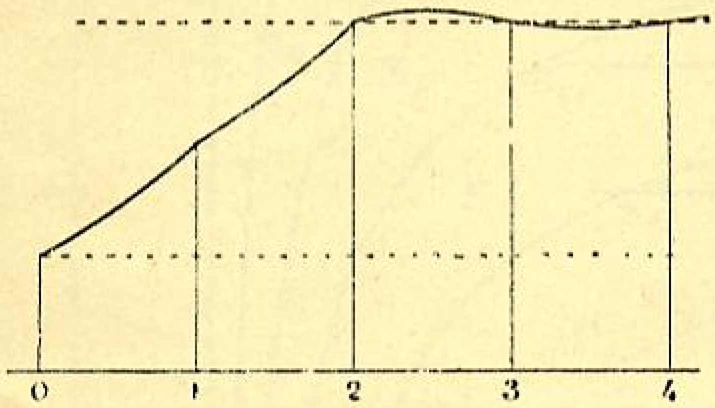


Fig. 15.ª (zona Σ_3)

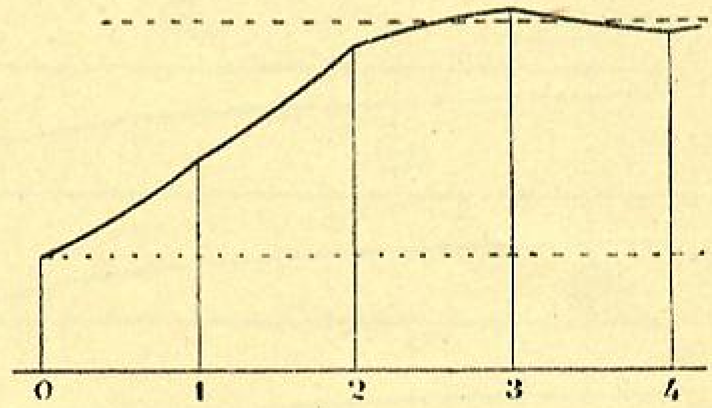


Fig. 16.ª (zone Θ_i)

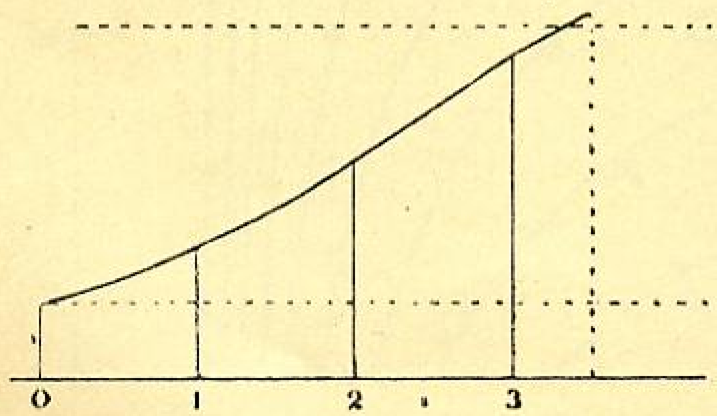


Fig. 17 (zona Θ)

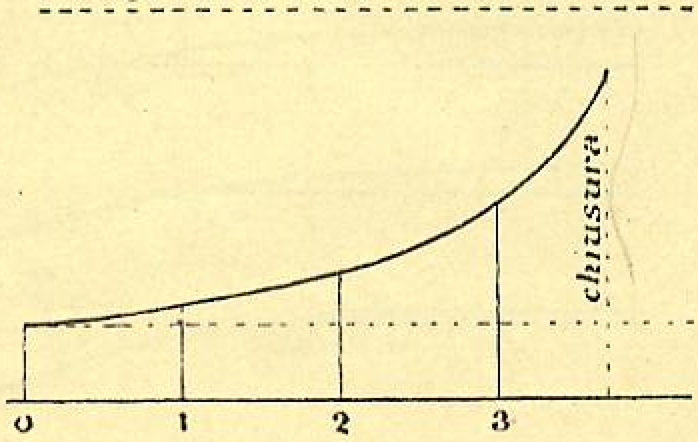


Fig. 18ª *Sinossi di classifica rispetto al carico massimo di ritmo intero in chiusura.*

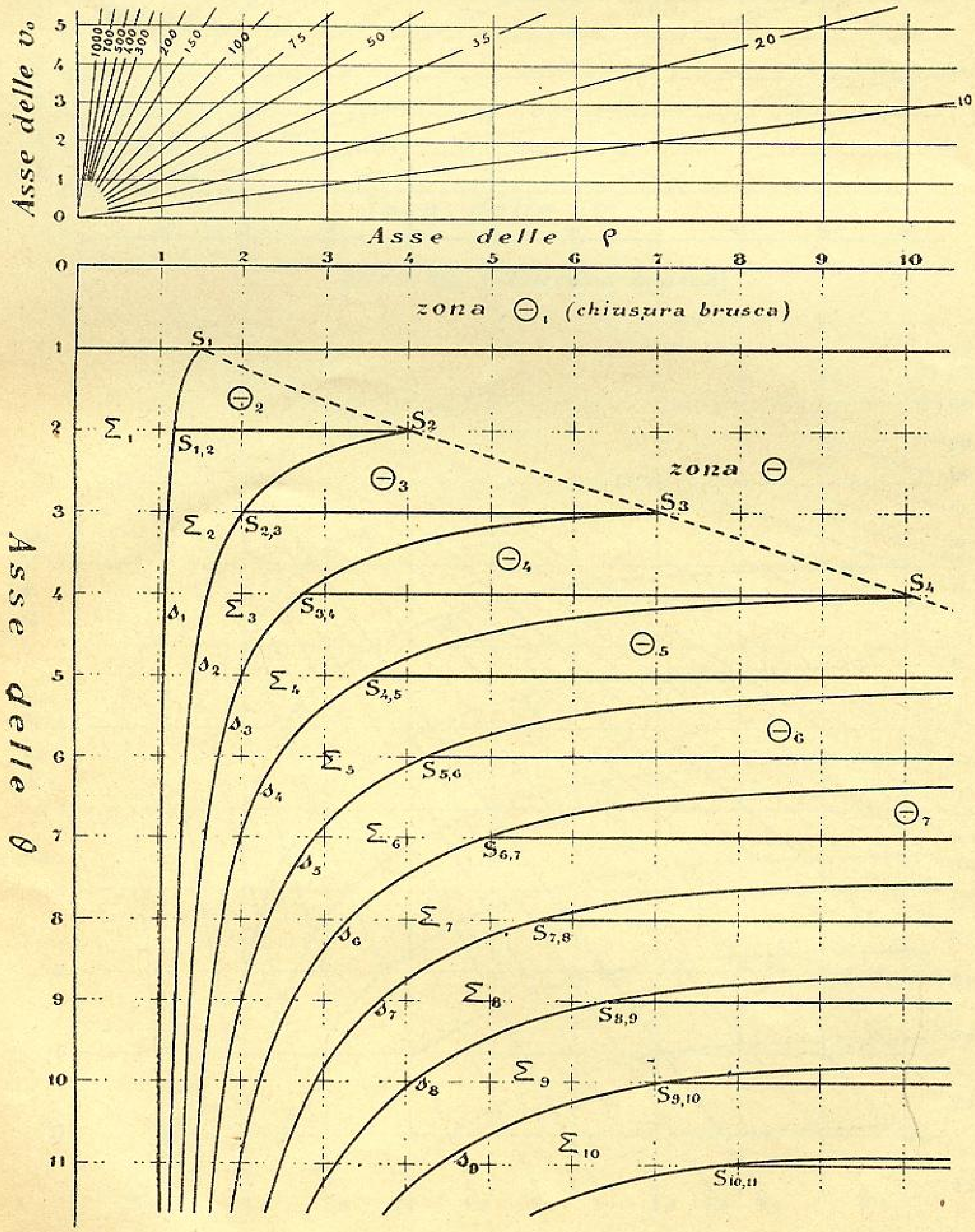
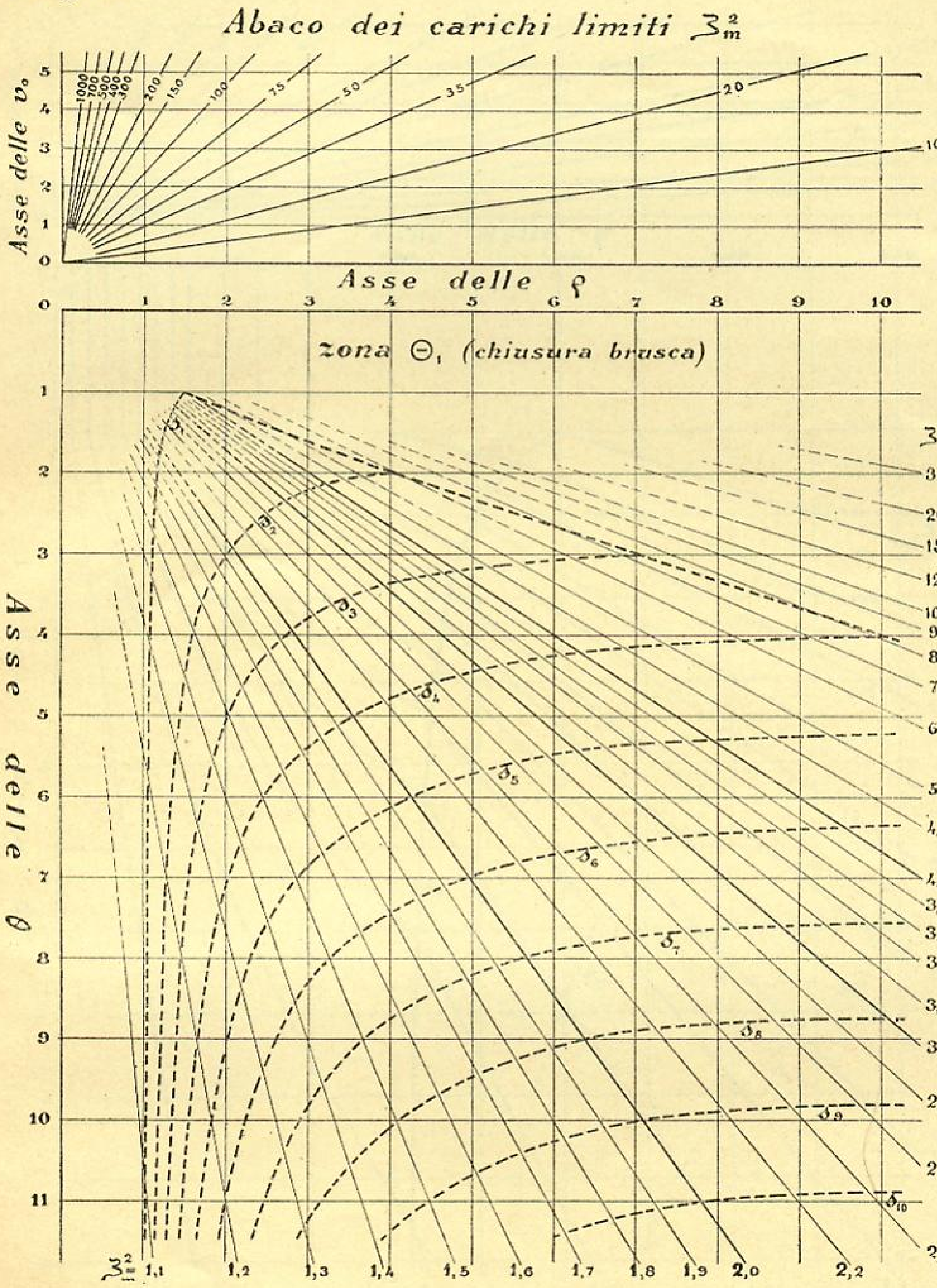


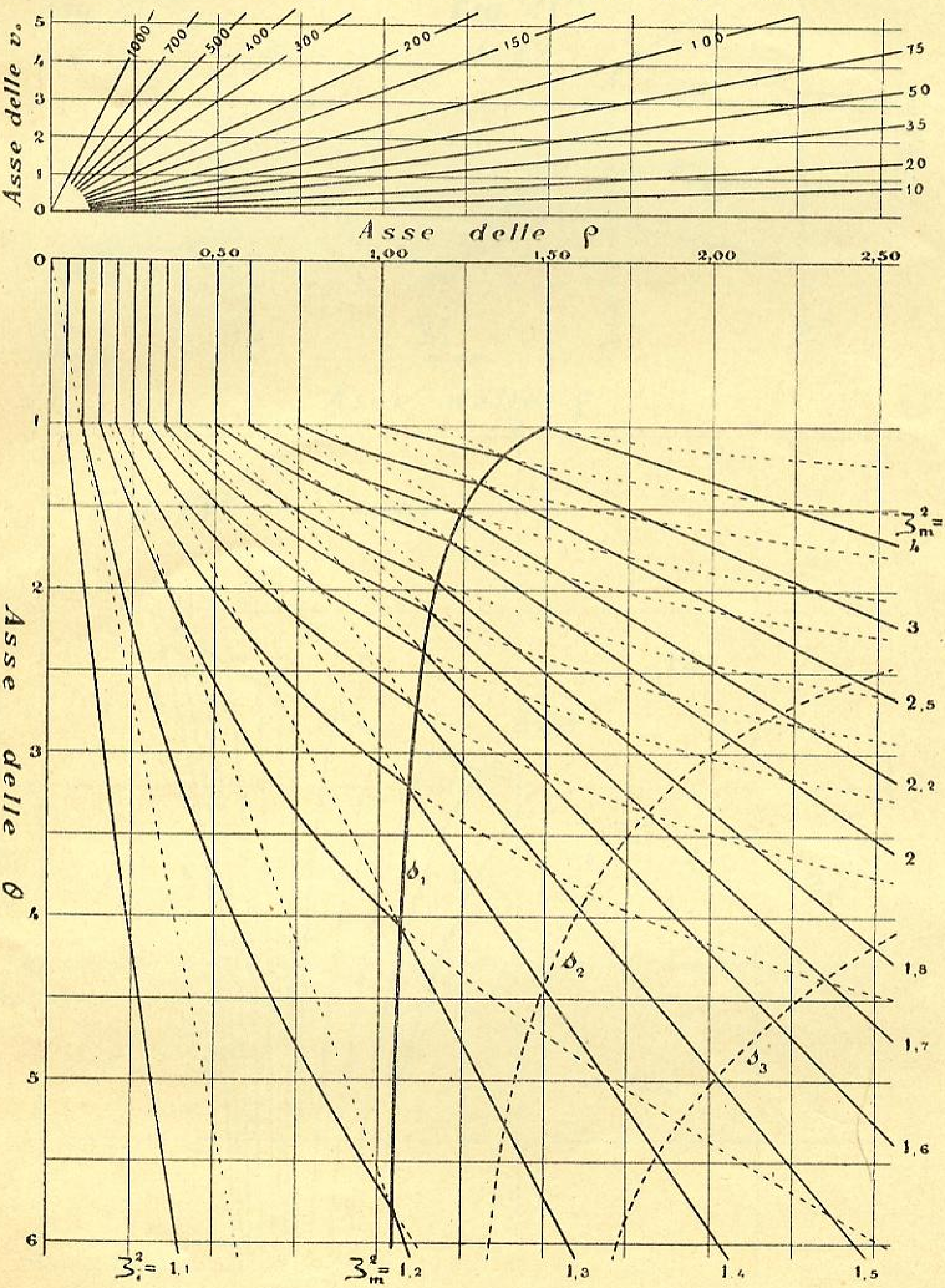
Fig. 19ª



Abaco dei carichi limiti $З^2_m$
Asse delle v_0
1000
700
500
400
300
200
150
100
75
50
35
20
10
Asse delle ρ
zona $Θ_1$ (chiusura brusca)
Asse delle θ
$З^2_m$
30
20
15
12
10
9
8
7
6
5
4,5
4,0
3,8
3,6
3,4
3,2
3,0
2,8
2,6
2,4
$З^2_m$= 1,1
1,2
1,3
1,4
1,5
1,6
1,7
1,8
1,9
2,0
2,2

Fig. 20.ª

Abaco dei carichi $\mathfrak{z}_1^2$ e $\mathfrak{z}_m^2$



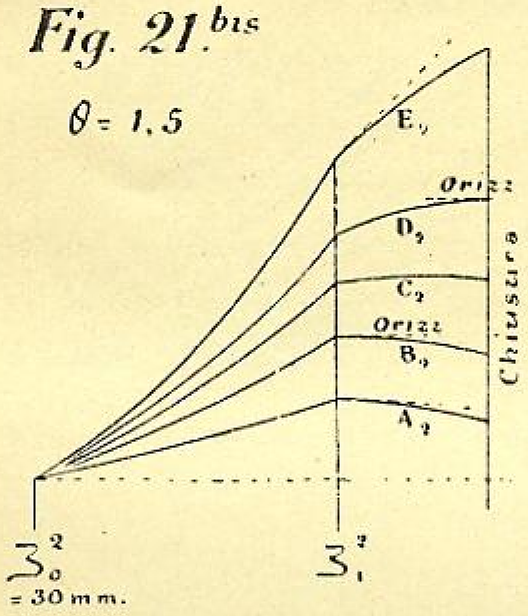


Fig. 21.bis

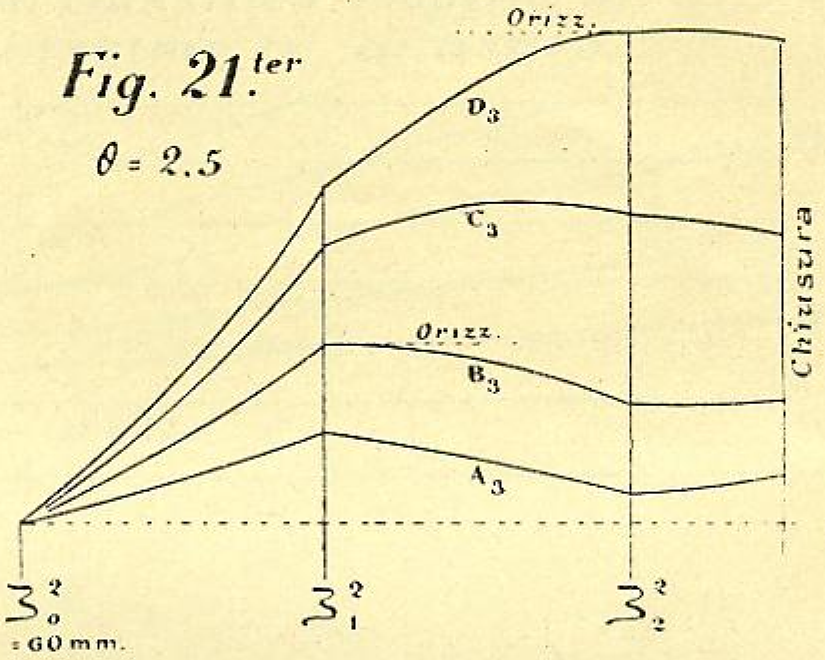


Fig. 21.ter

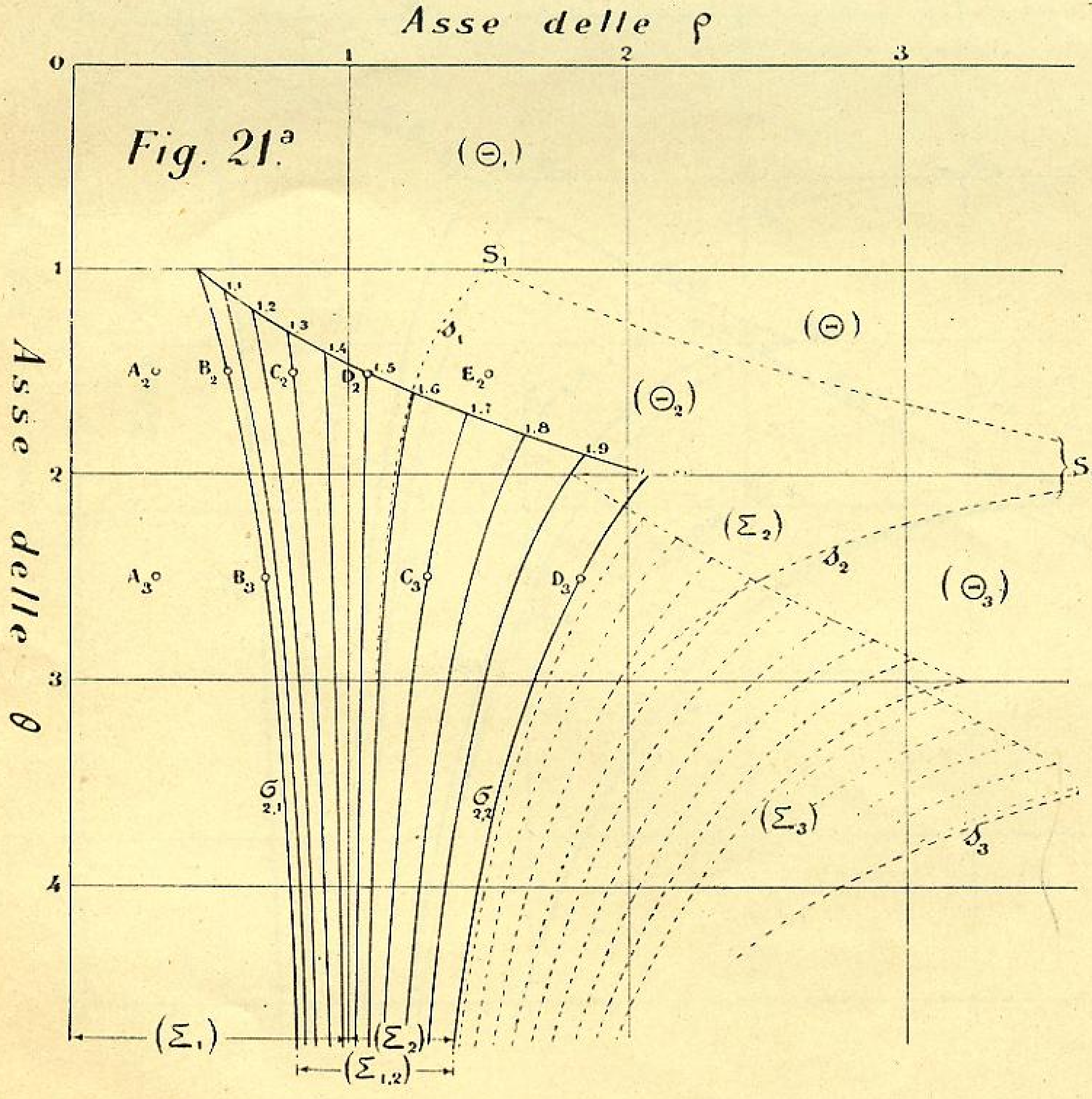


Fig. 21.a

Fig. 22ª *Sinossi di classifica rispetto al carico massimo in chiusura.*

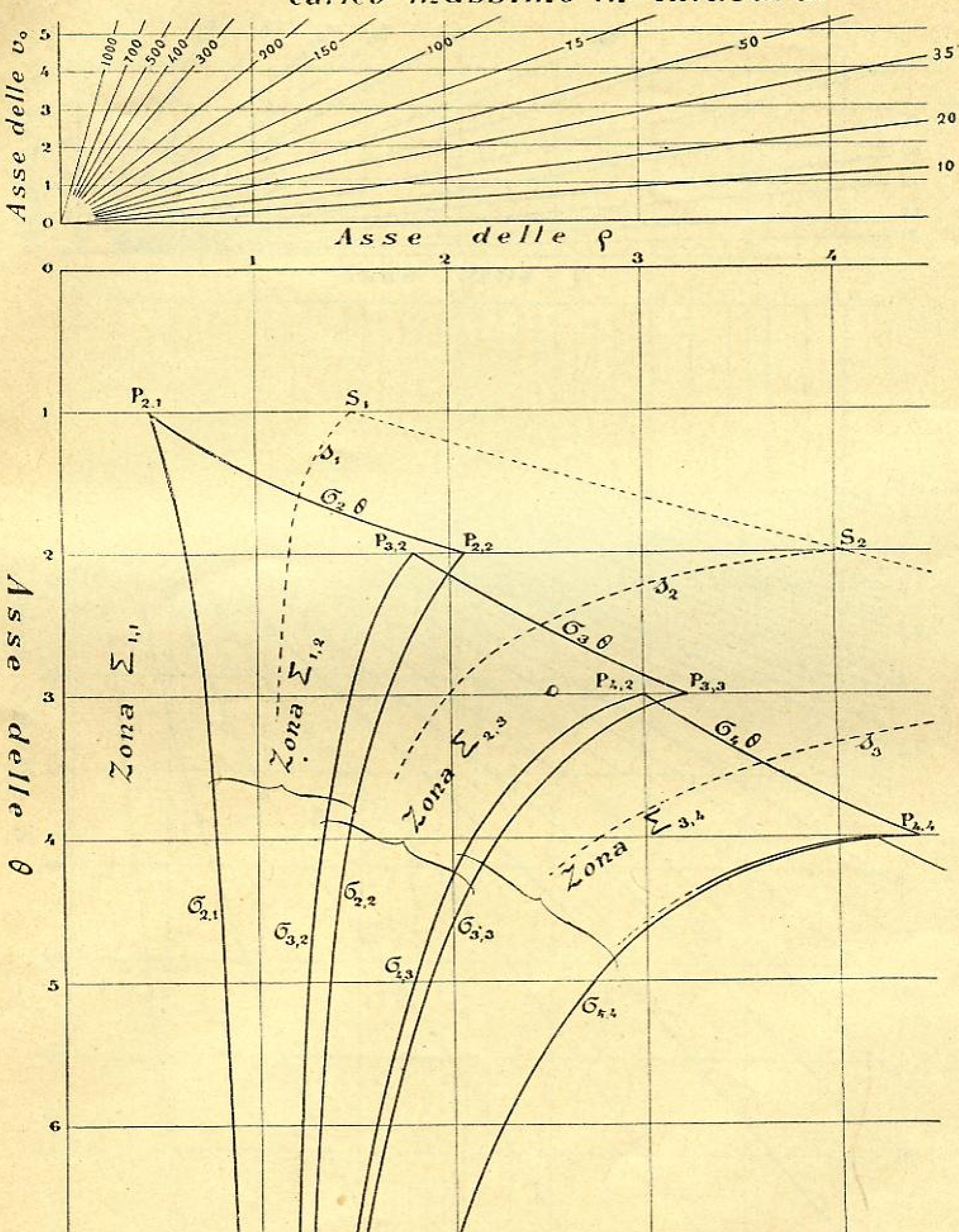
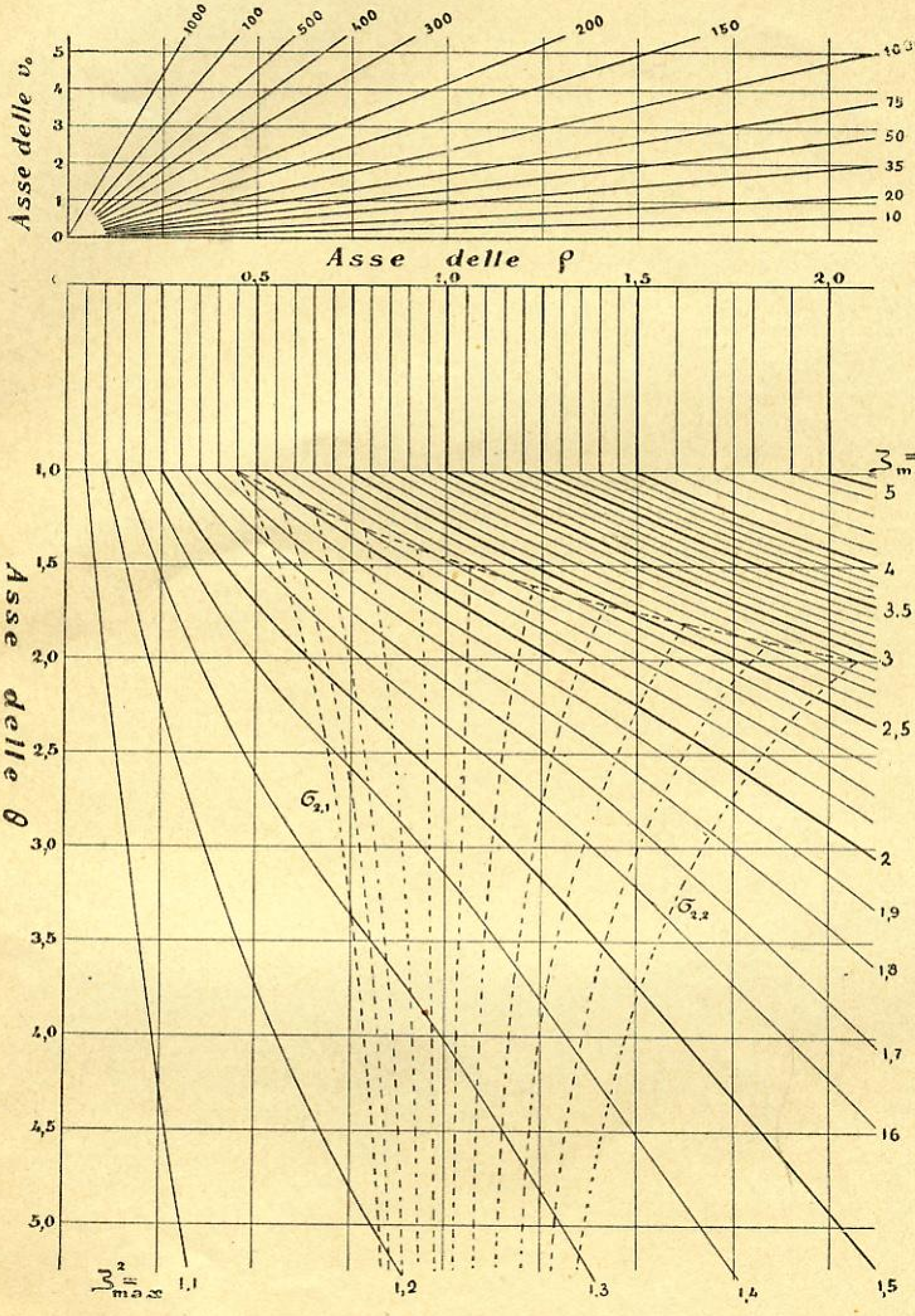


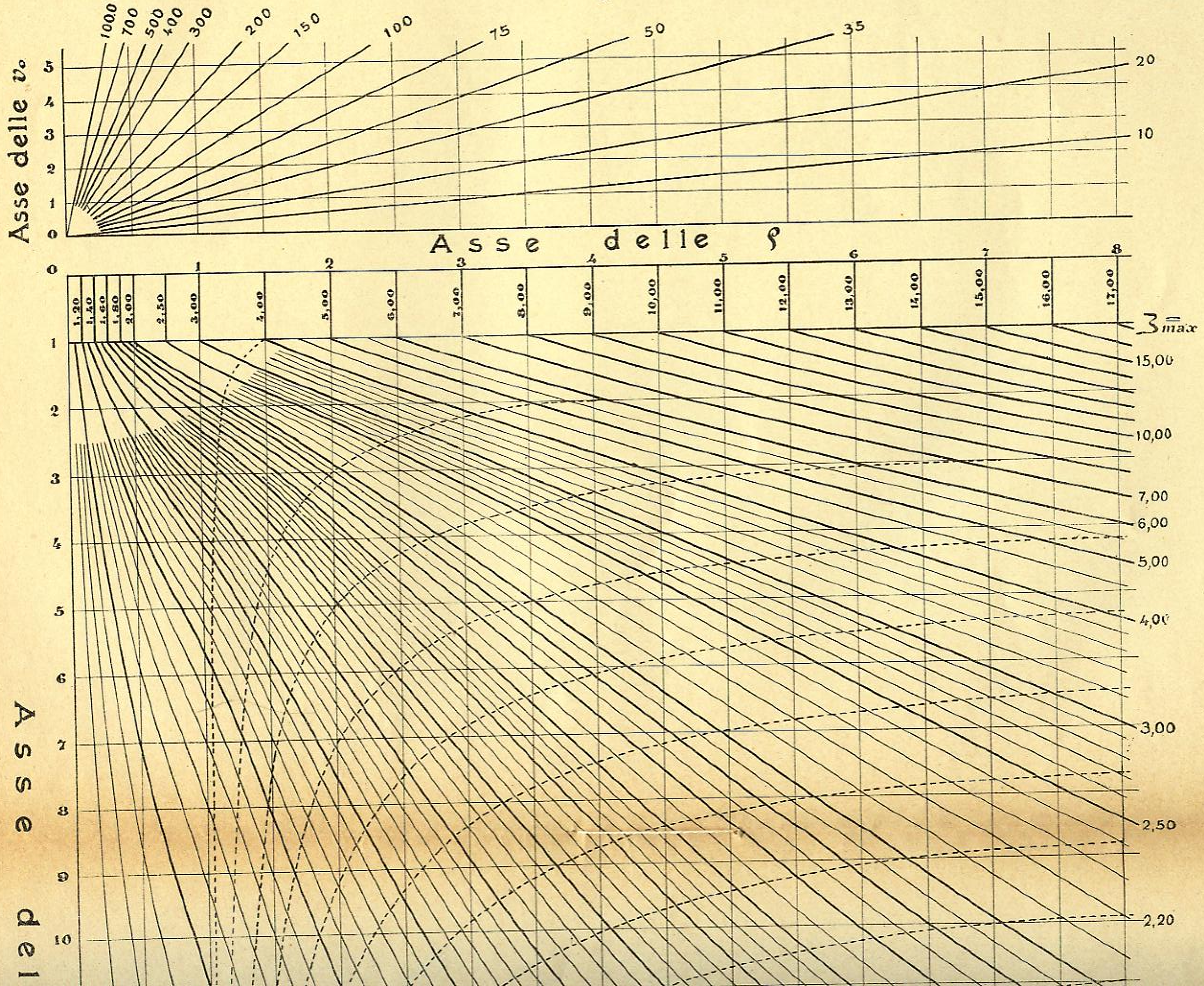
Fig. 23

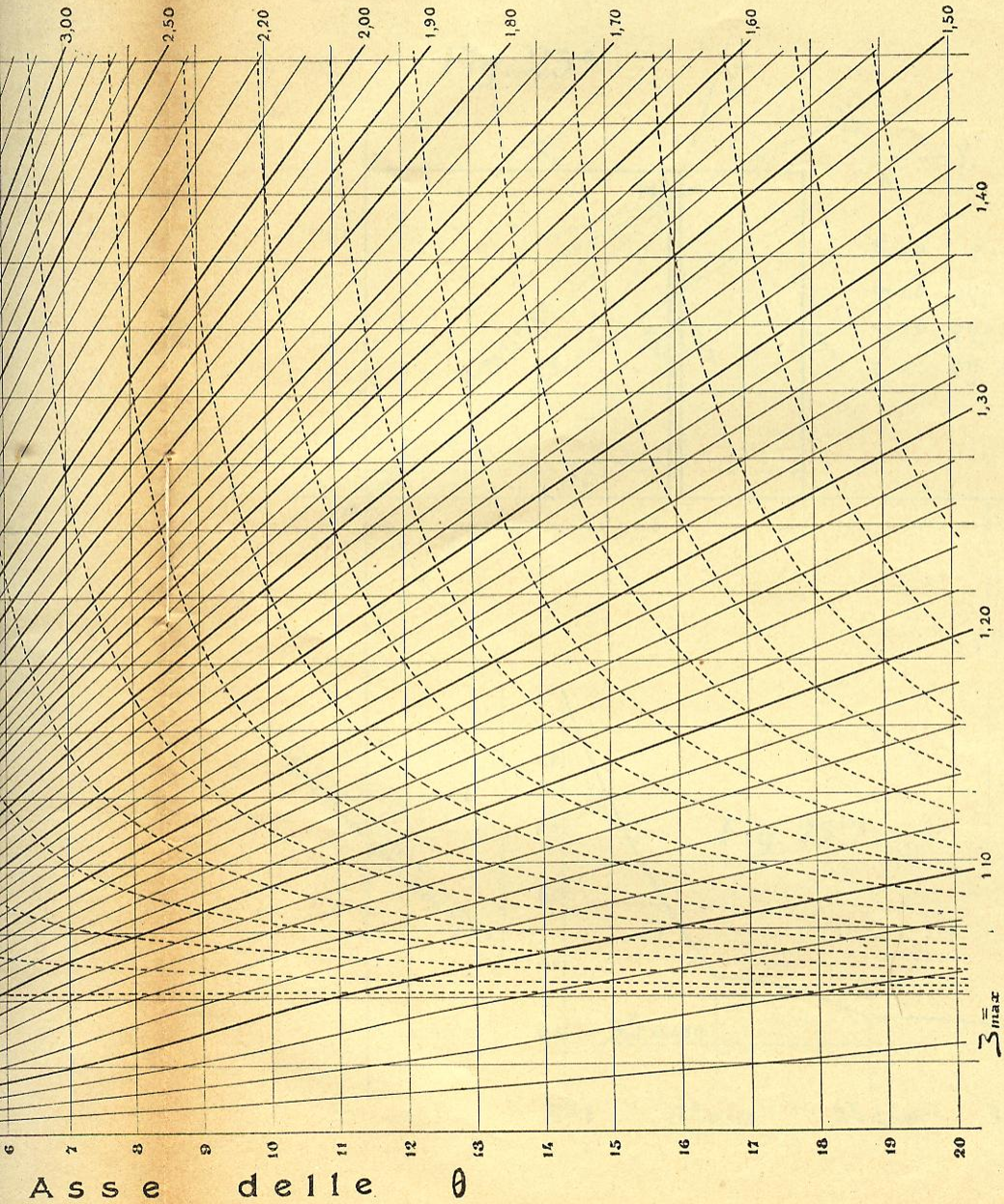


Asse delle v_0
1000
100
500
400
300
200
150
100
75
50
35
20
10
Asse delle ρ
0,5
1,0
1,5
2,0
Asse delle θ
1,0
1,5
2,0
2,5
3,0
3,5
4,0
4,5
5,0
ζ max =
5
4
3.5
3
2,5
2
1,9
1,8
1,7
16
σ2,1
σ2,2
ζ² max =
1,1
1,2
1,3
1,4
1,5

Fig. 24

Abaco generale





3,00
2,50
2,20
2,00
1,90
1,80
1,70
1,60
1,50
1,40
1,30
1,20
1 10
$\bar{\bar{z}}_{max}$
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
Asse delle θ

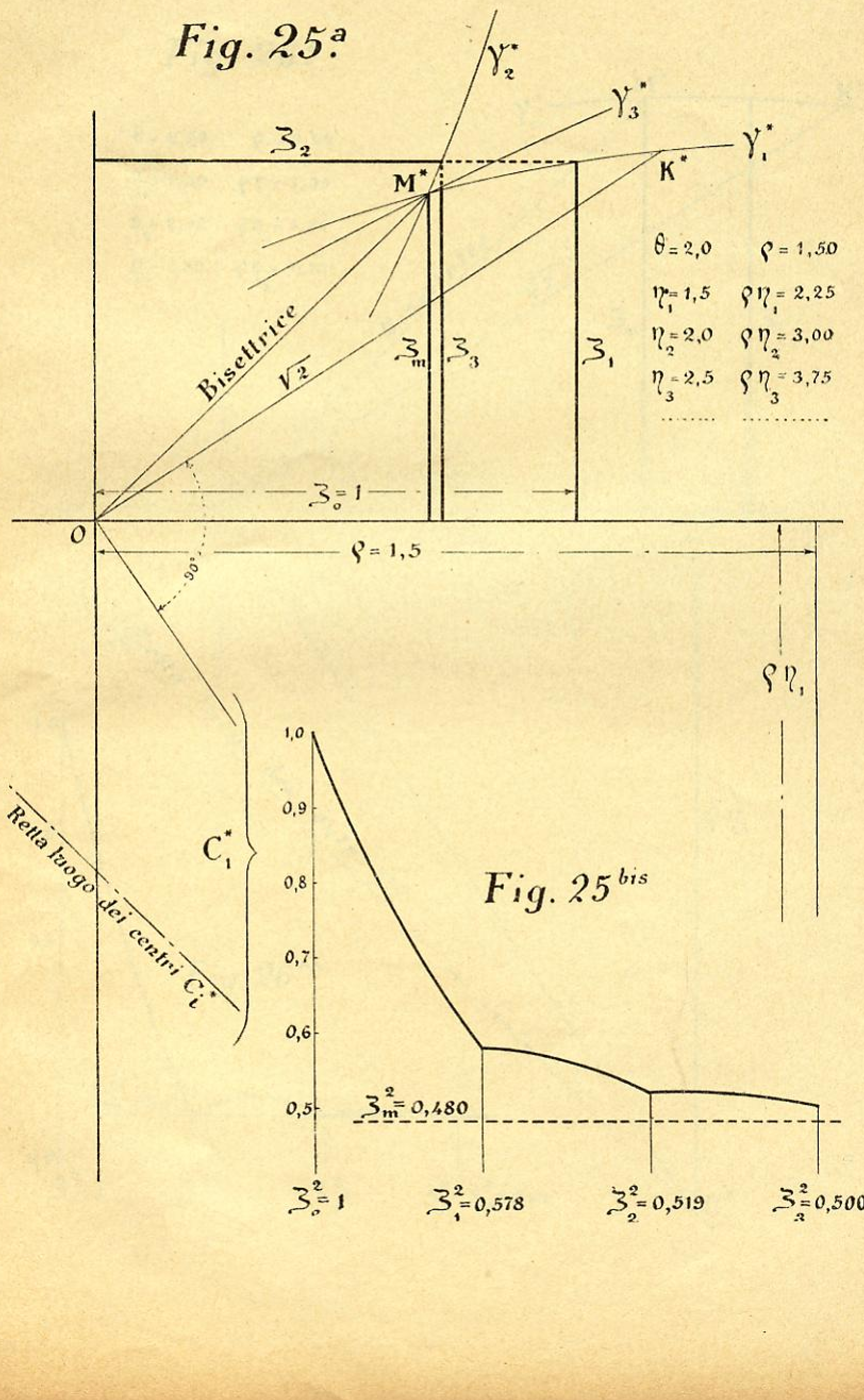


Fig. 25ª

Fig. 25 bis

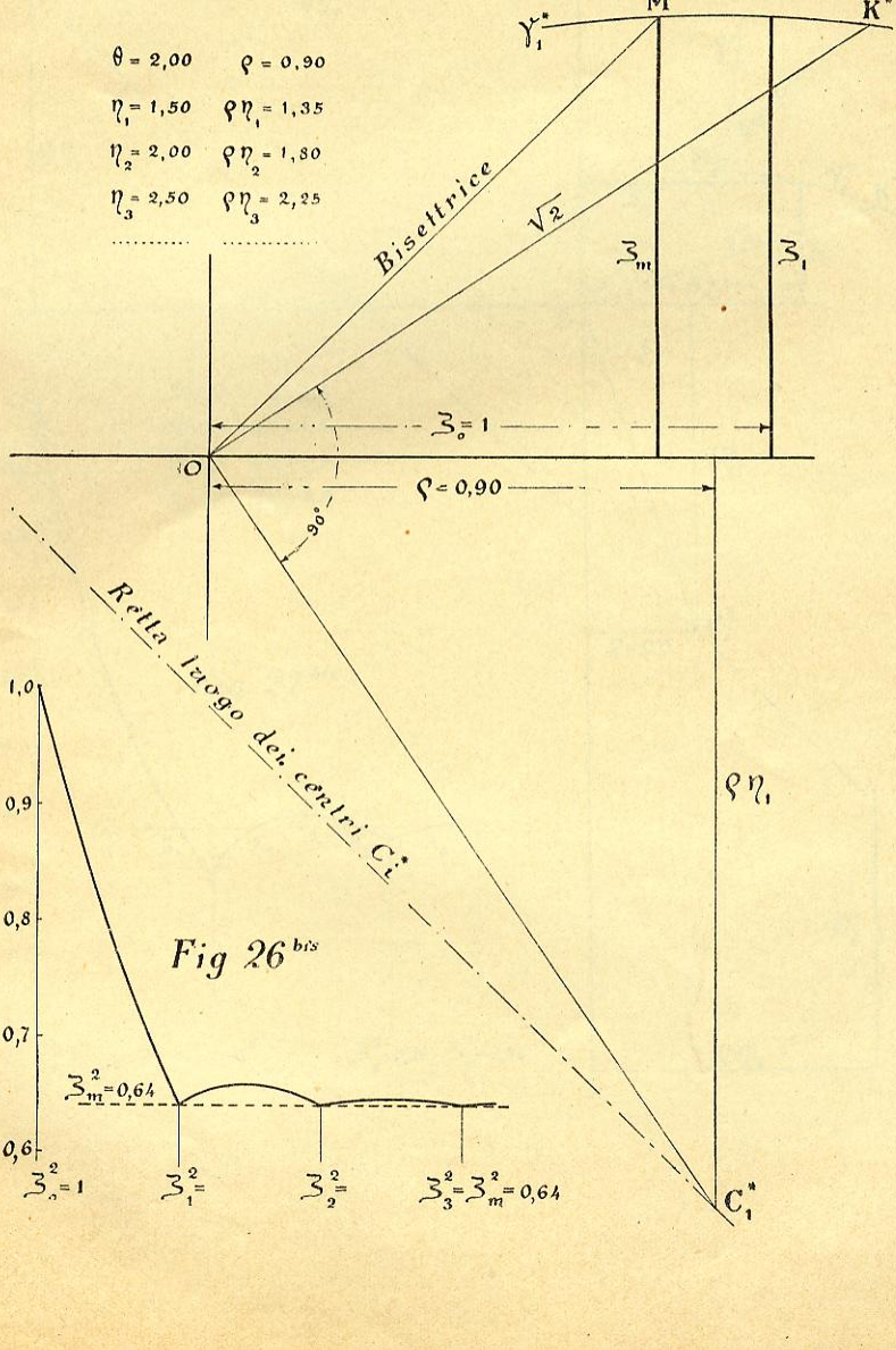


Fig. 26ª.

Fig 26 bis

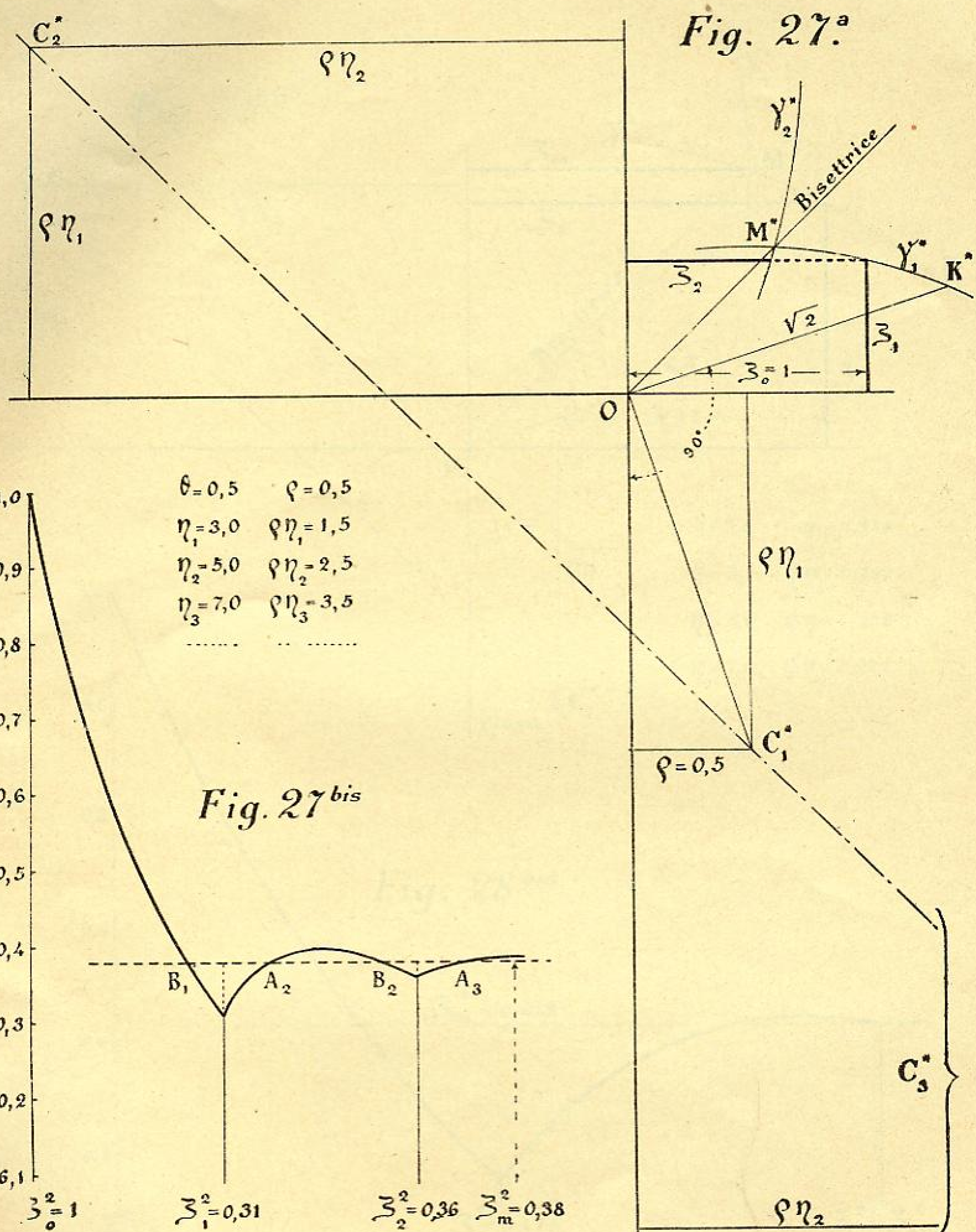


Fig. 27.ª

Fig. 27 bis

Fig. 28ª

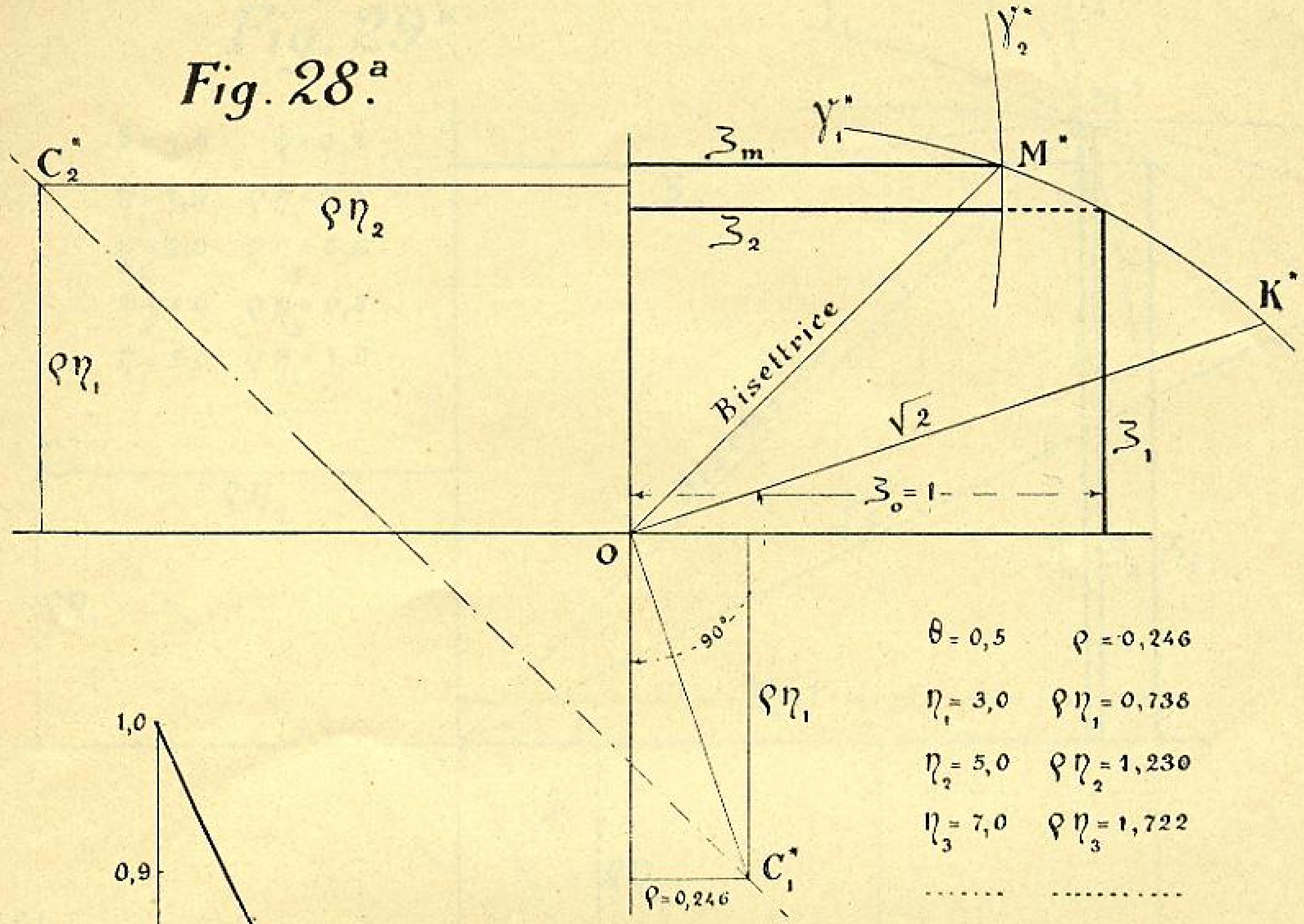
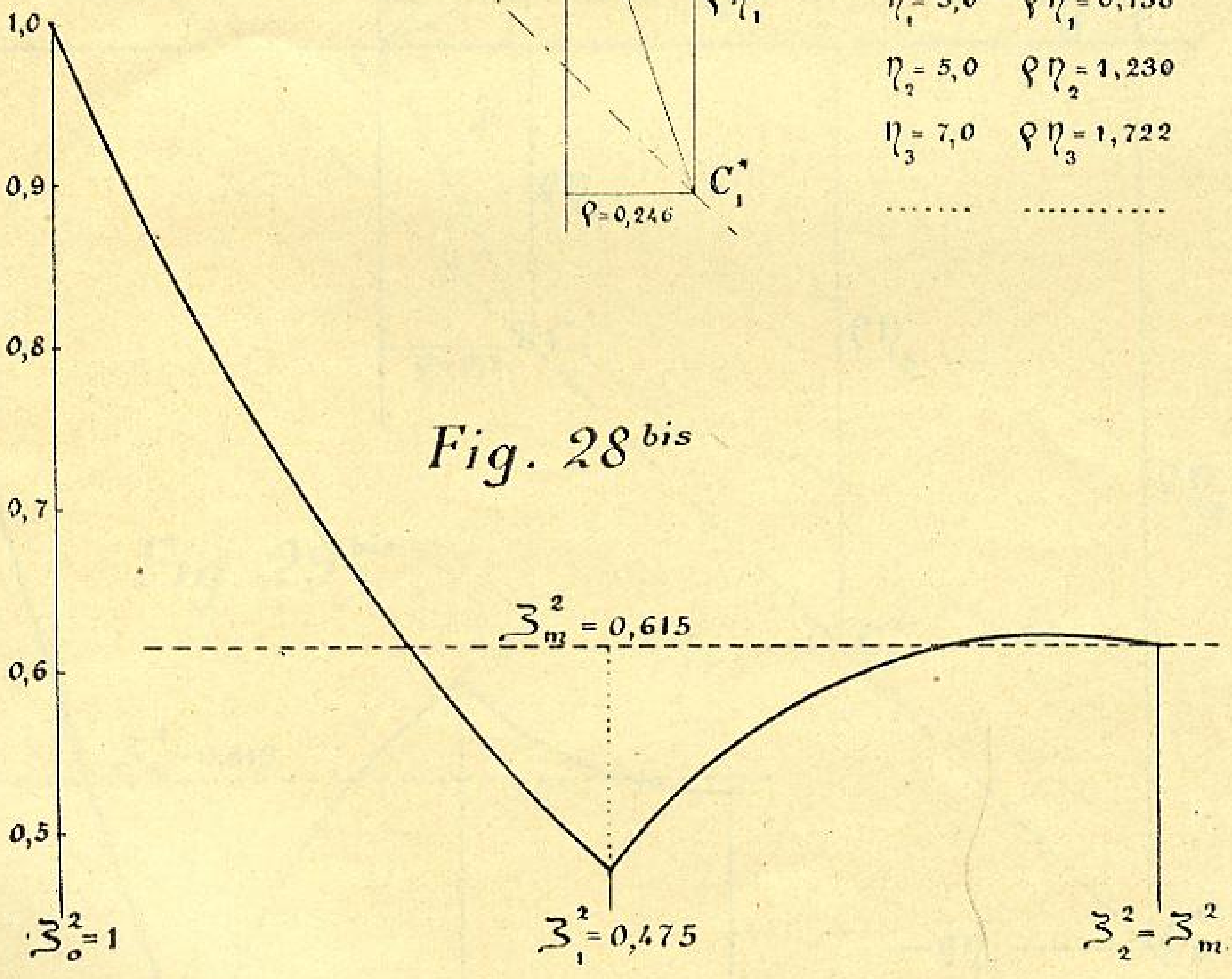


Fig. 28 bis



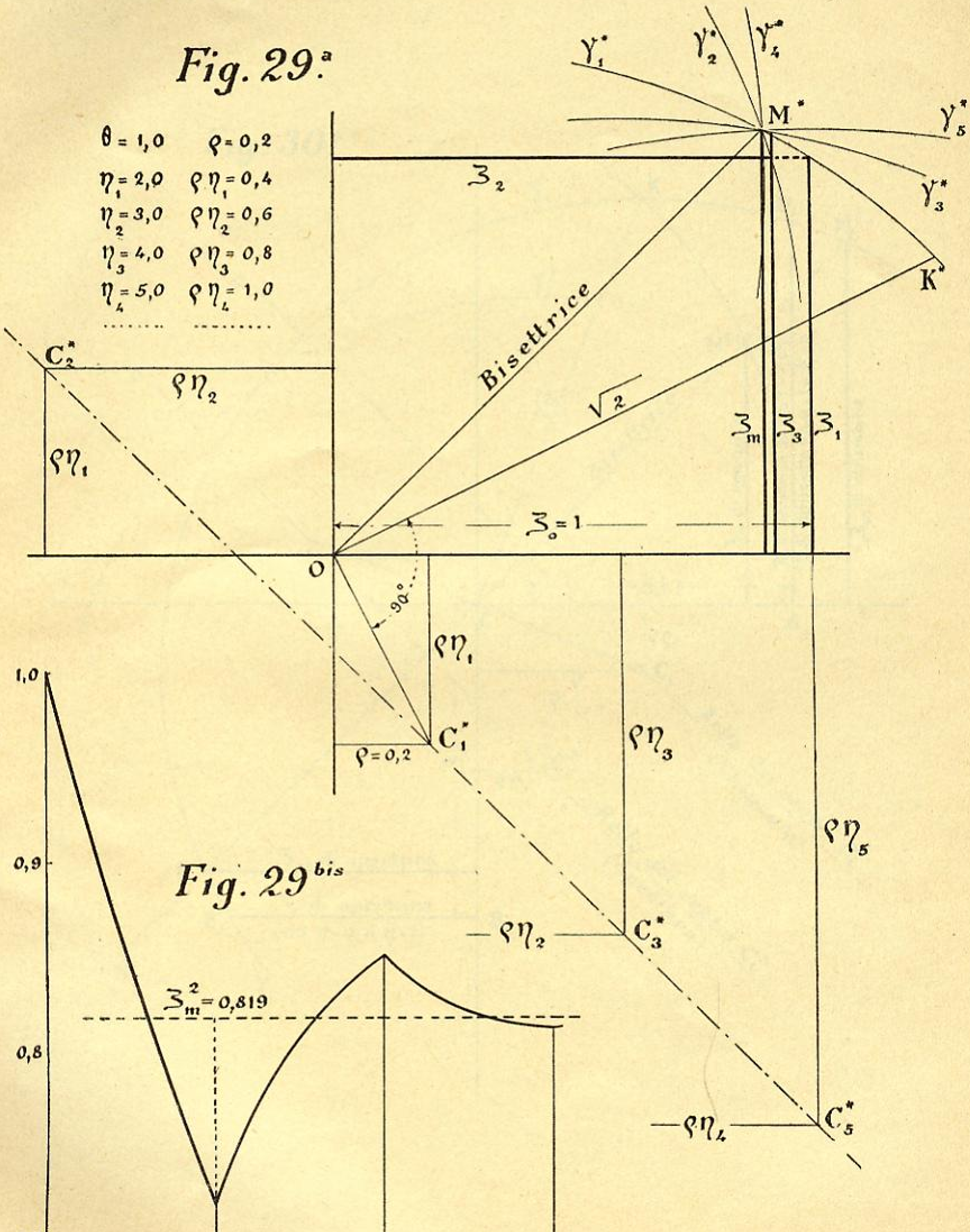
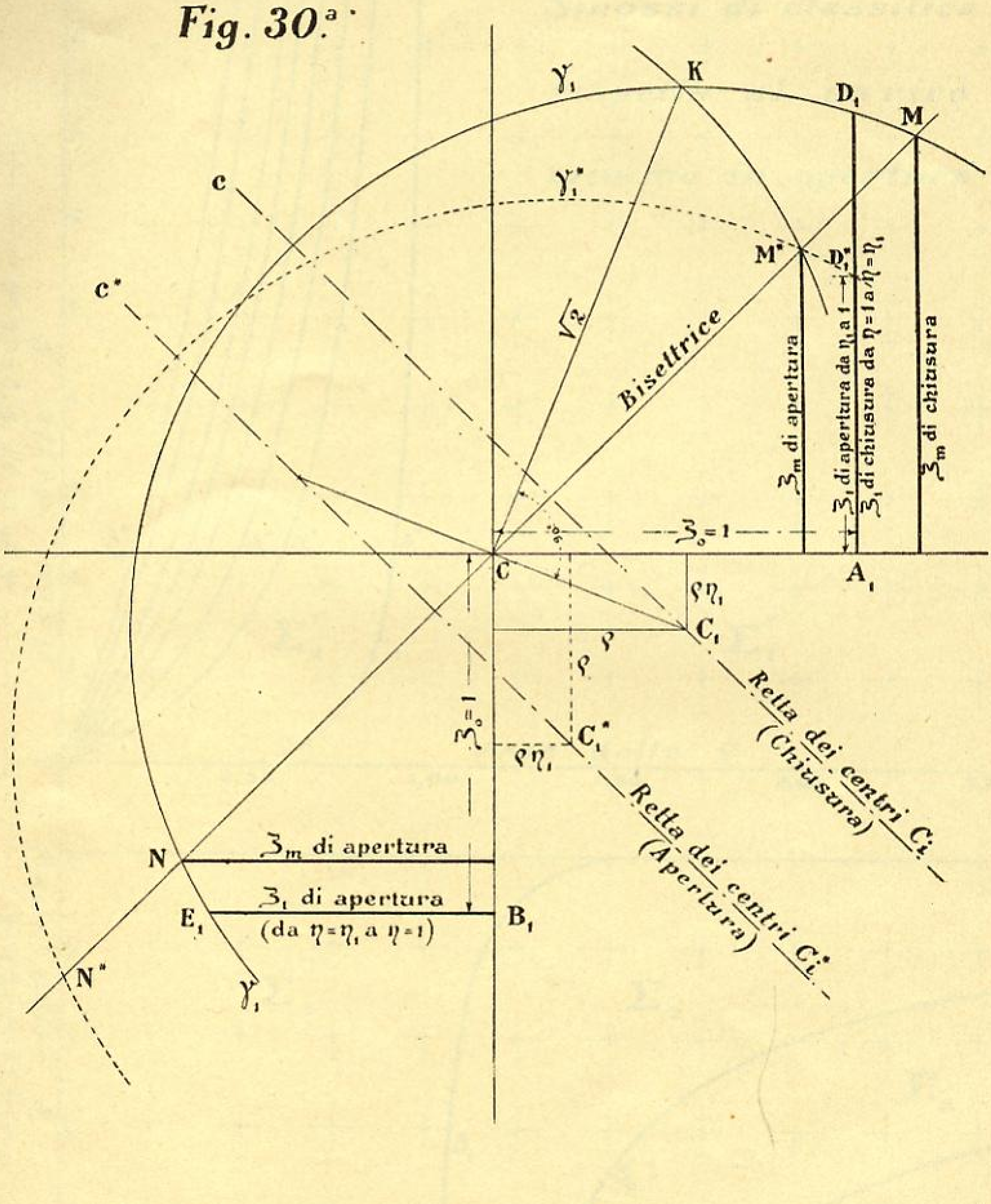


Fig. 29a

Fig. 29bis

Fig. 30.ª



K
D₁
M
γ₁
γ₁*
c
c*
M*
D₁*
√2
Bisettrice
ℨm di apertura
ℨ₁ di apertura da η₁ a 1
ℨ₁ di chiusura da η = 1 a η = η₁
ℨm di chiusura
ℨ₀ = 1
C
A₁
ρη₁
C₁
ρ
C₁*
ℨ₀ = 1
Retta dei centri C₁ (Chiusura)
Retta dei centri C₁* (Apertura)
N
ℨm di apertura
ℨ₁ di apertura (da η = η₁ a η = 1)
E₁
B₁
N*
γ₁

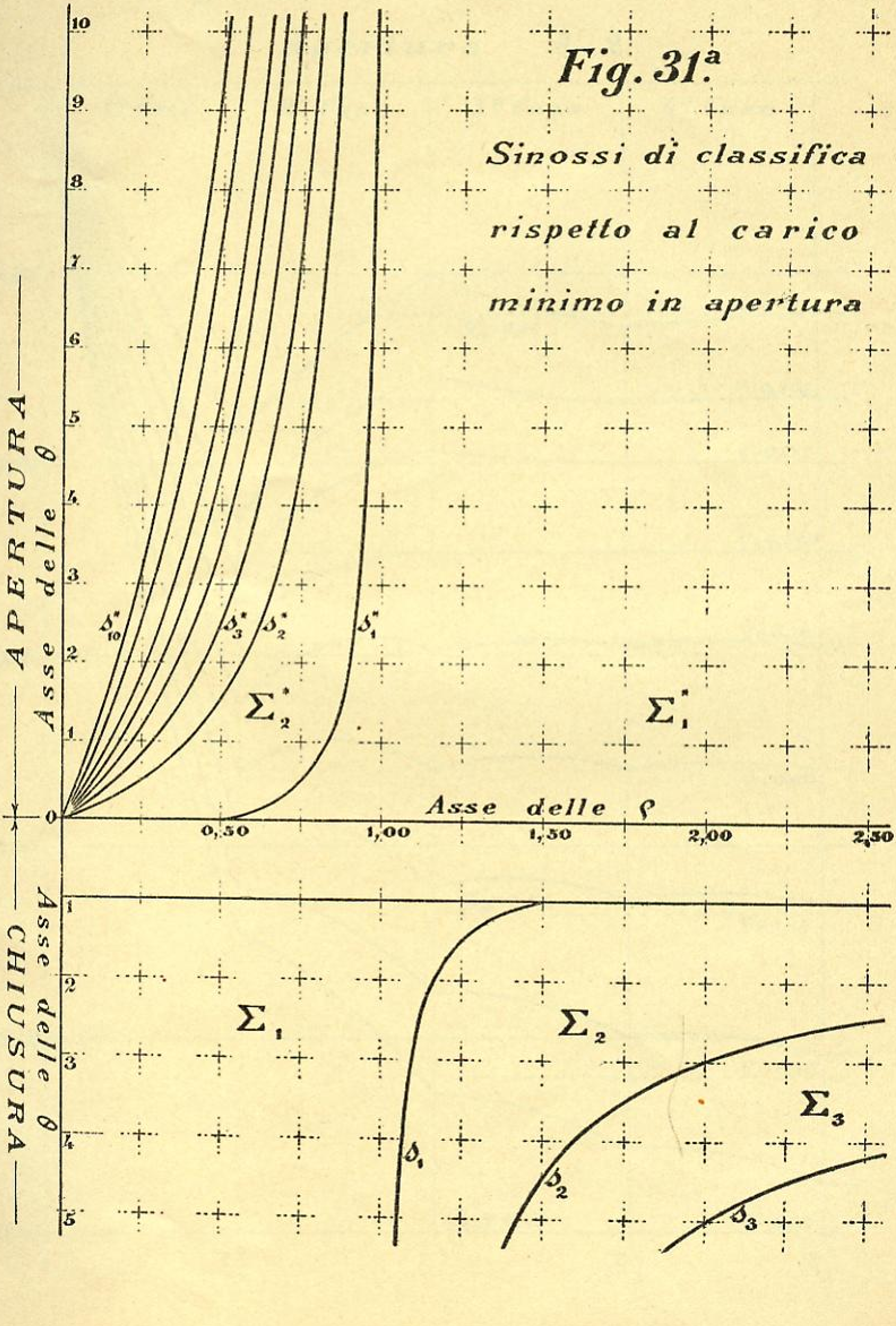


Fig. 31ª

Sinossi di classifica rispetto al carico minimo in apertura

Fig. 32.ª

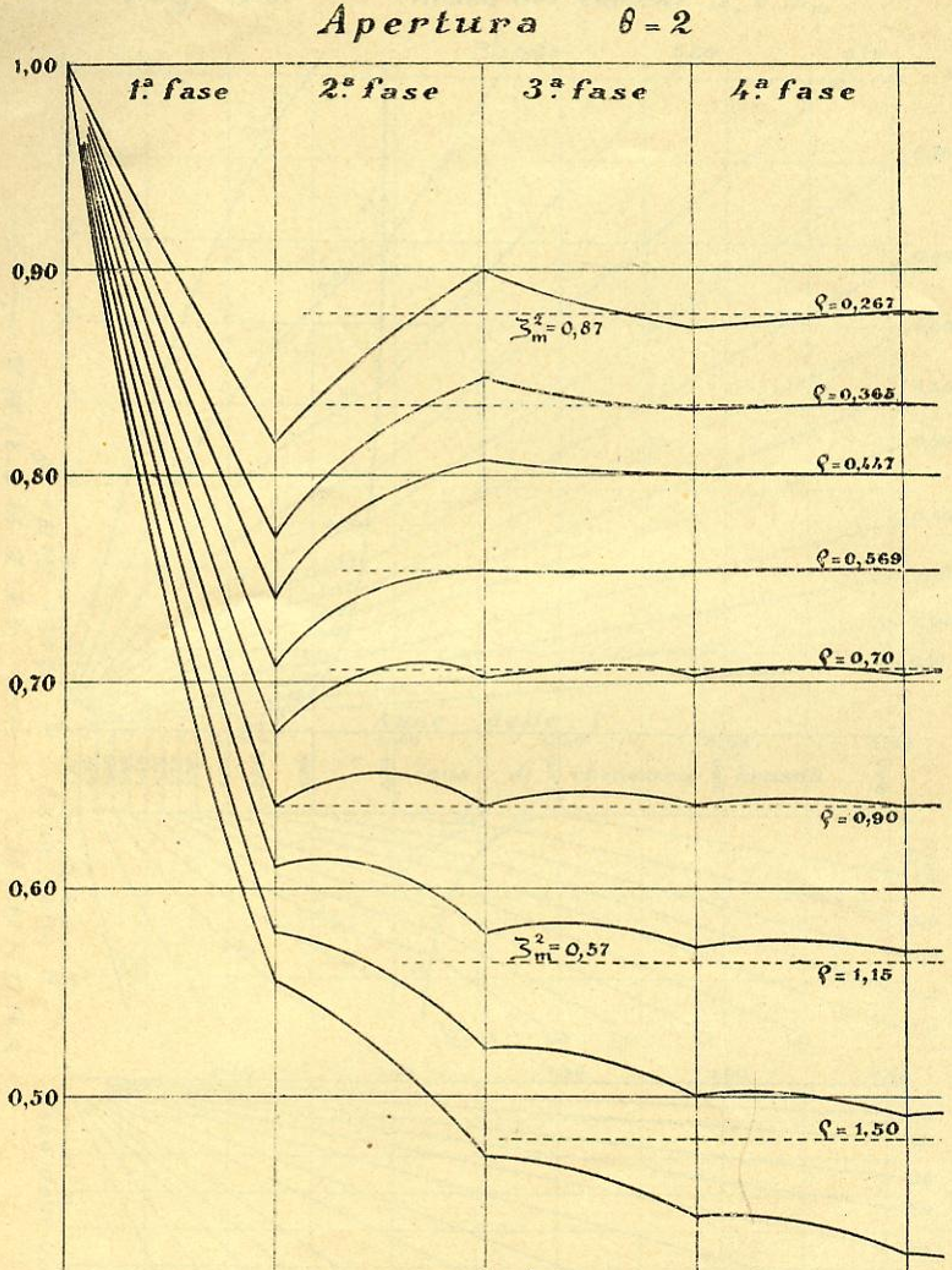
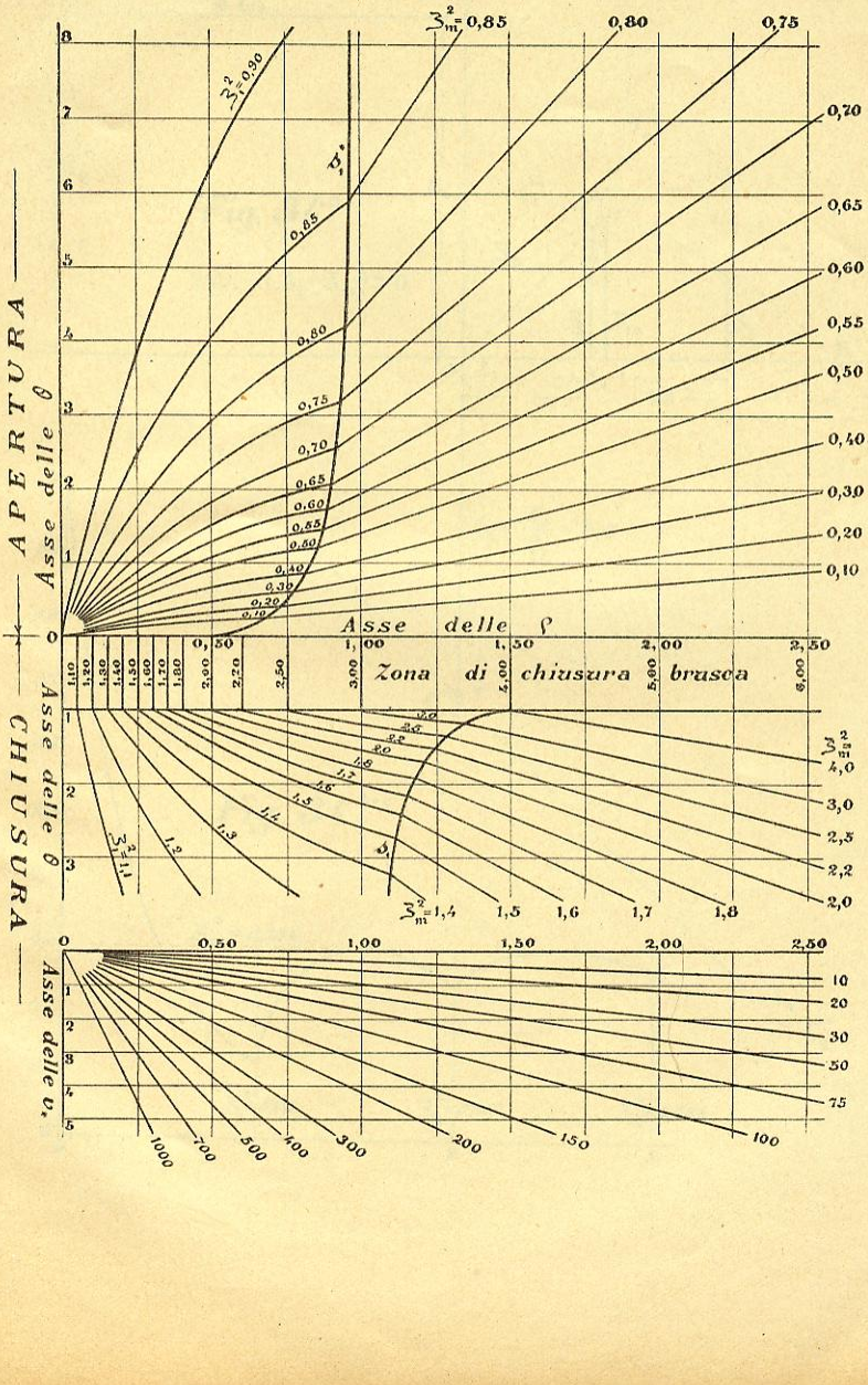

Apertura θ = 2
1,00
0,90
0,80
0,70
0,60
0,50
0,40
1ª fase
2ª fase
3ª fase
4ª fase
ρ = 0,267
ρ = 0,365
ρ = 0,447
ρ = 0,569
ρ = 0,70
ρ = 0,90
ρ = 1,15
ρ = 1,50
ρ = 1,90
$\mathfrak{z}_m^2 = 0,87$
$\mathfrak{z}_m^2 = 0,57$
$\mathfrak{z}_m^2 = 0,40$
1
$\mathfrak{z}_1^2$
$\mathfrak{z}_2^2$
$\mathfrak{z}_3^2$
$\mathfrak{z}_4^2$

Fig. 33ª *Abaco dei carichi* ζ_1^2 *e* ζ_m^2



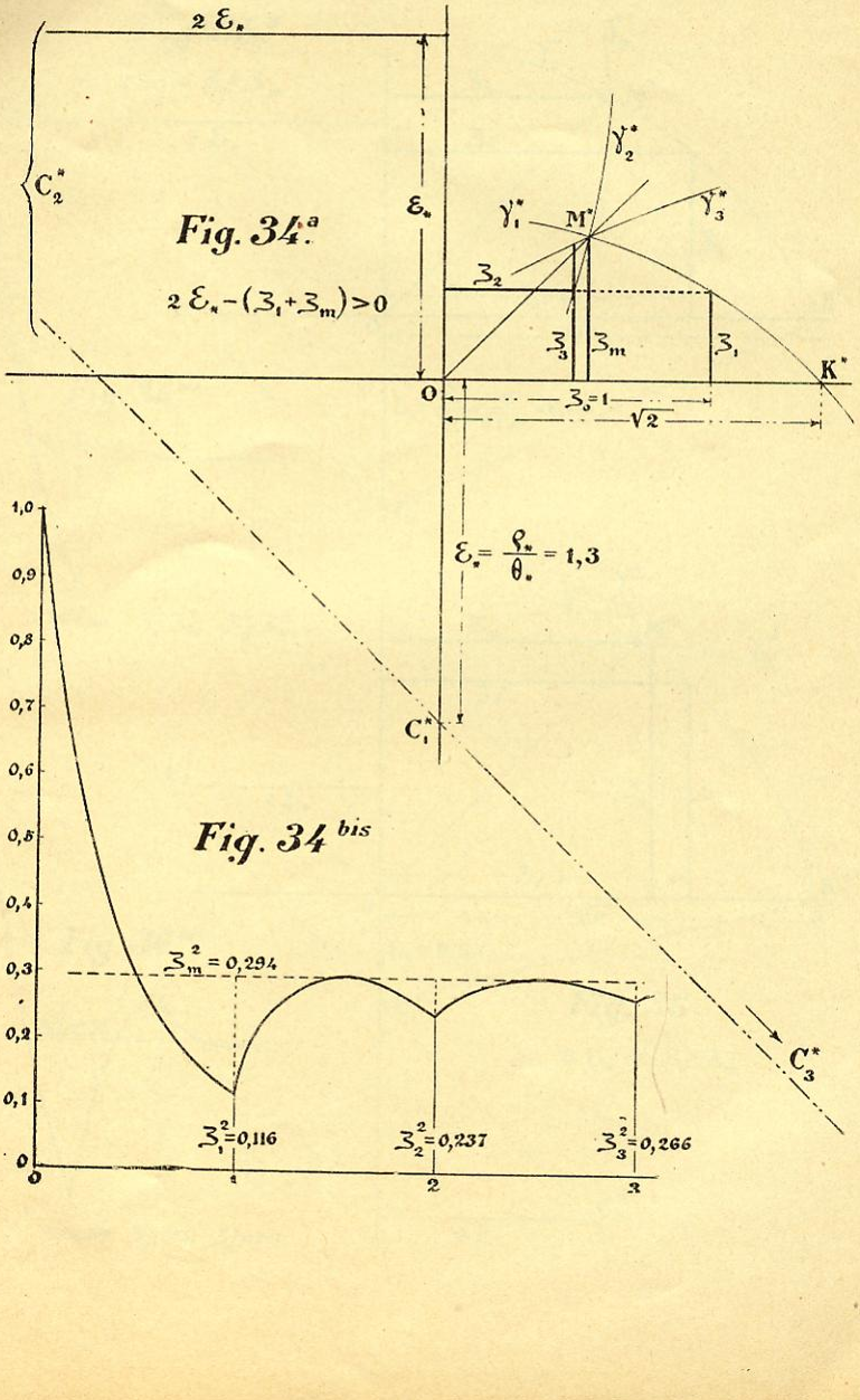


Fig. 34.a

Fig. 34 bis

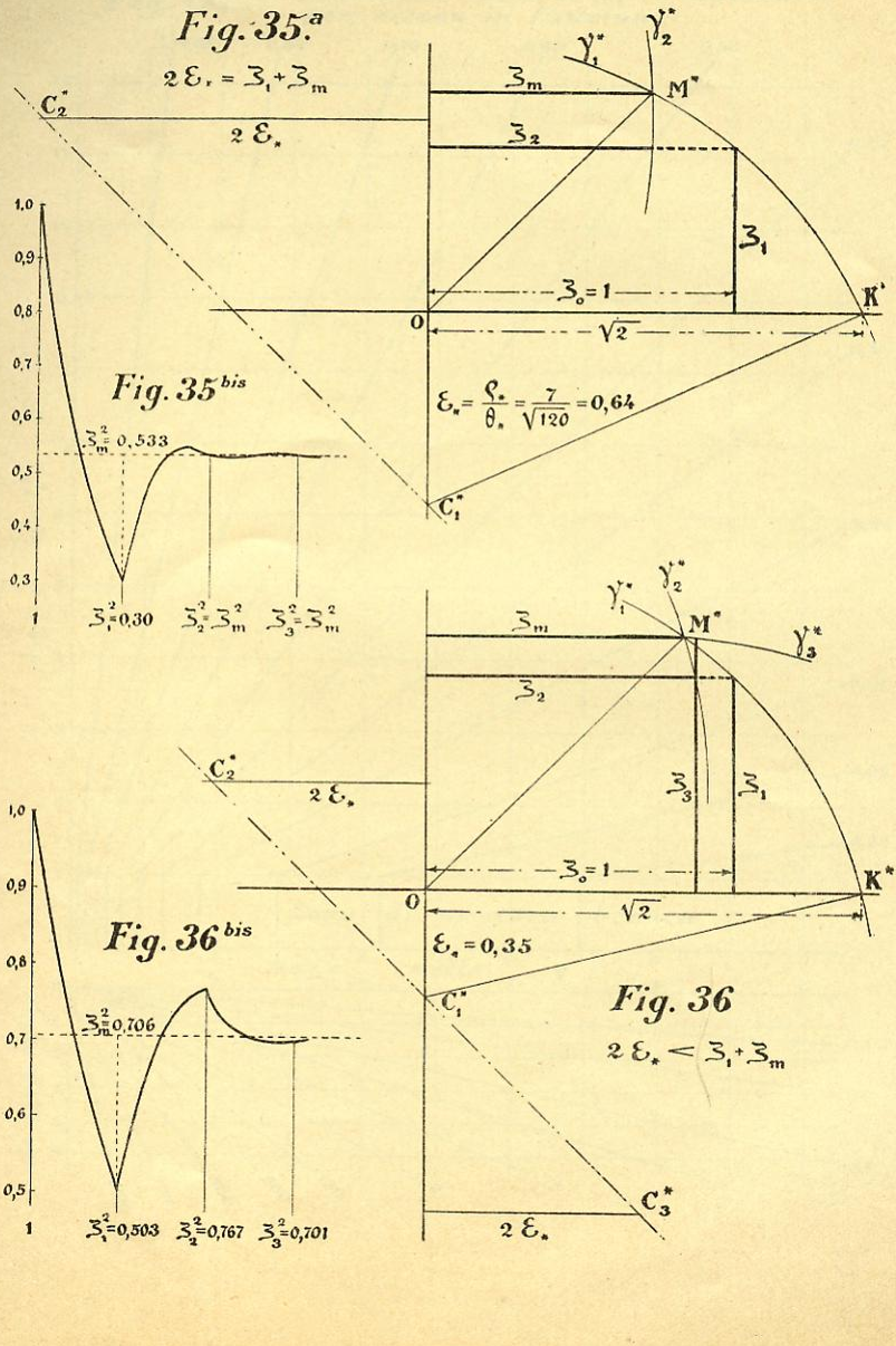


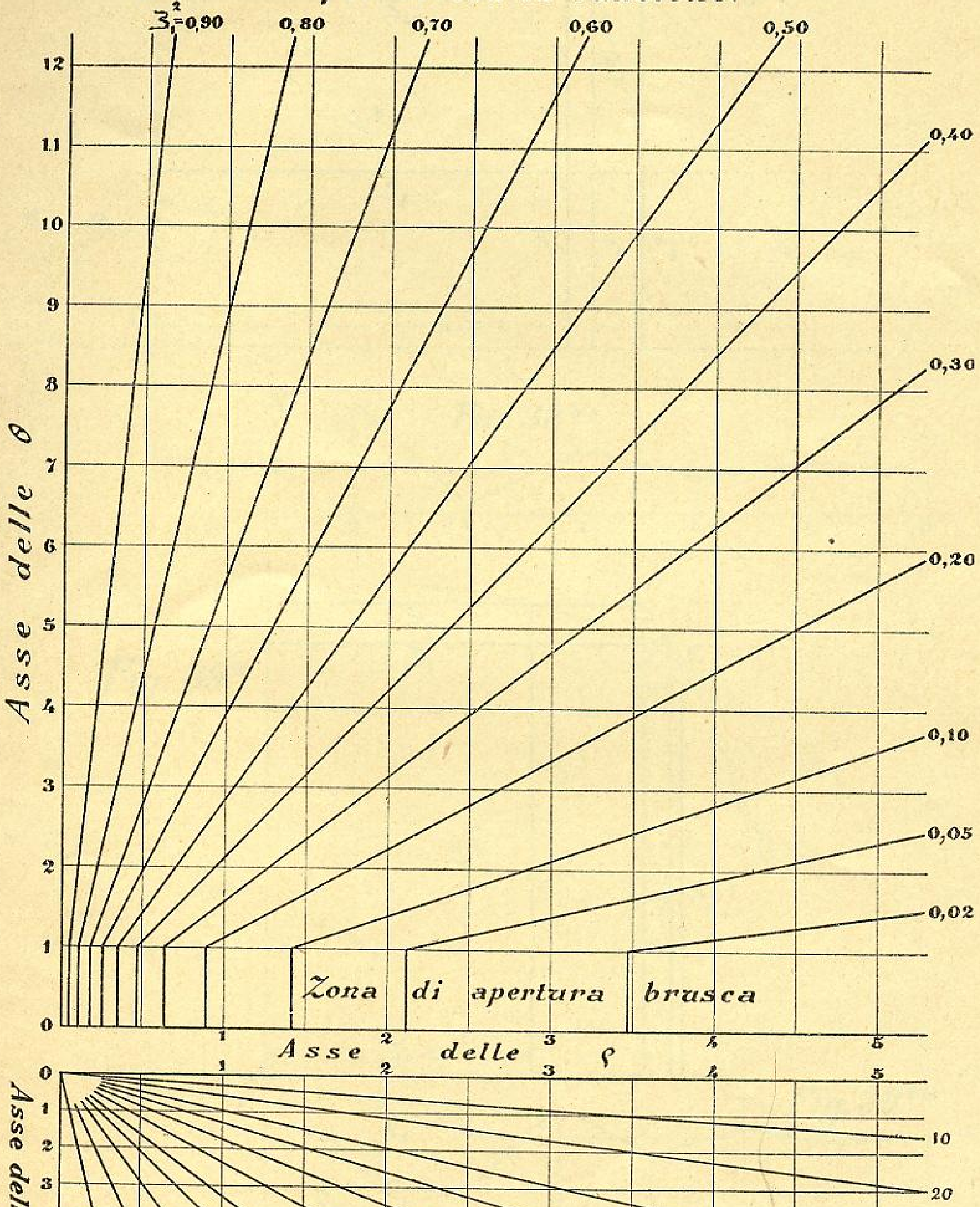
Fig. 35.a $2\varepsilon_* = \zeta_1 + \zeta_m$

Fig. 35 bis

Fig. 36 $2\varepsilon_* < \zeta_1 + \zeta_m$

Fig. 36 bis

Fig. 37.ª Abaco dei carichi $\mathfrak{Z}_1^2$ in apertura per messa in funzione.



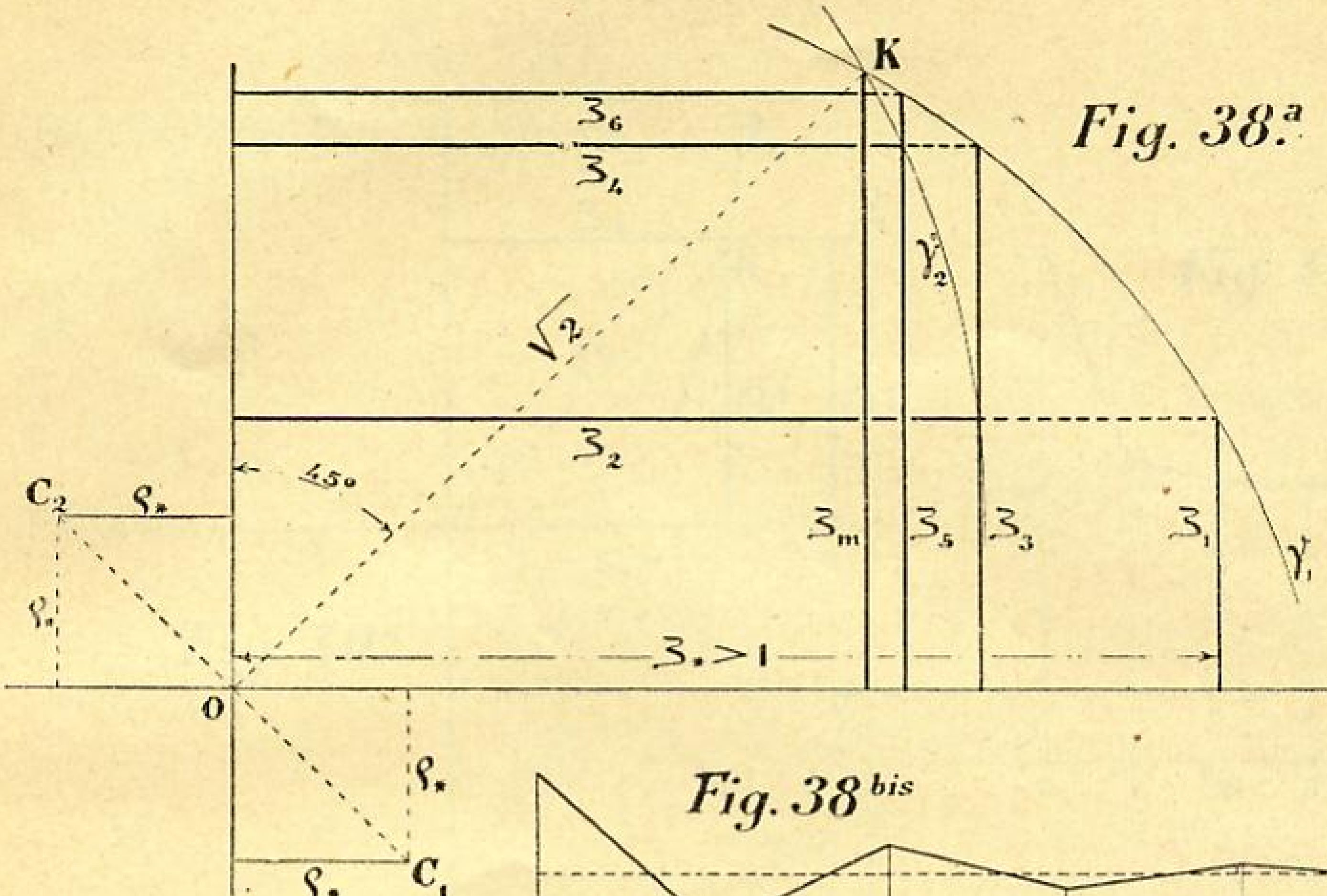


Fig. 38.a

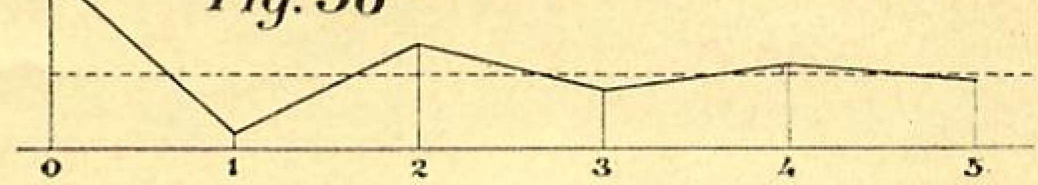


Fig. 38 bis

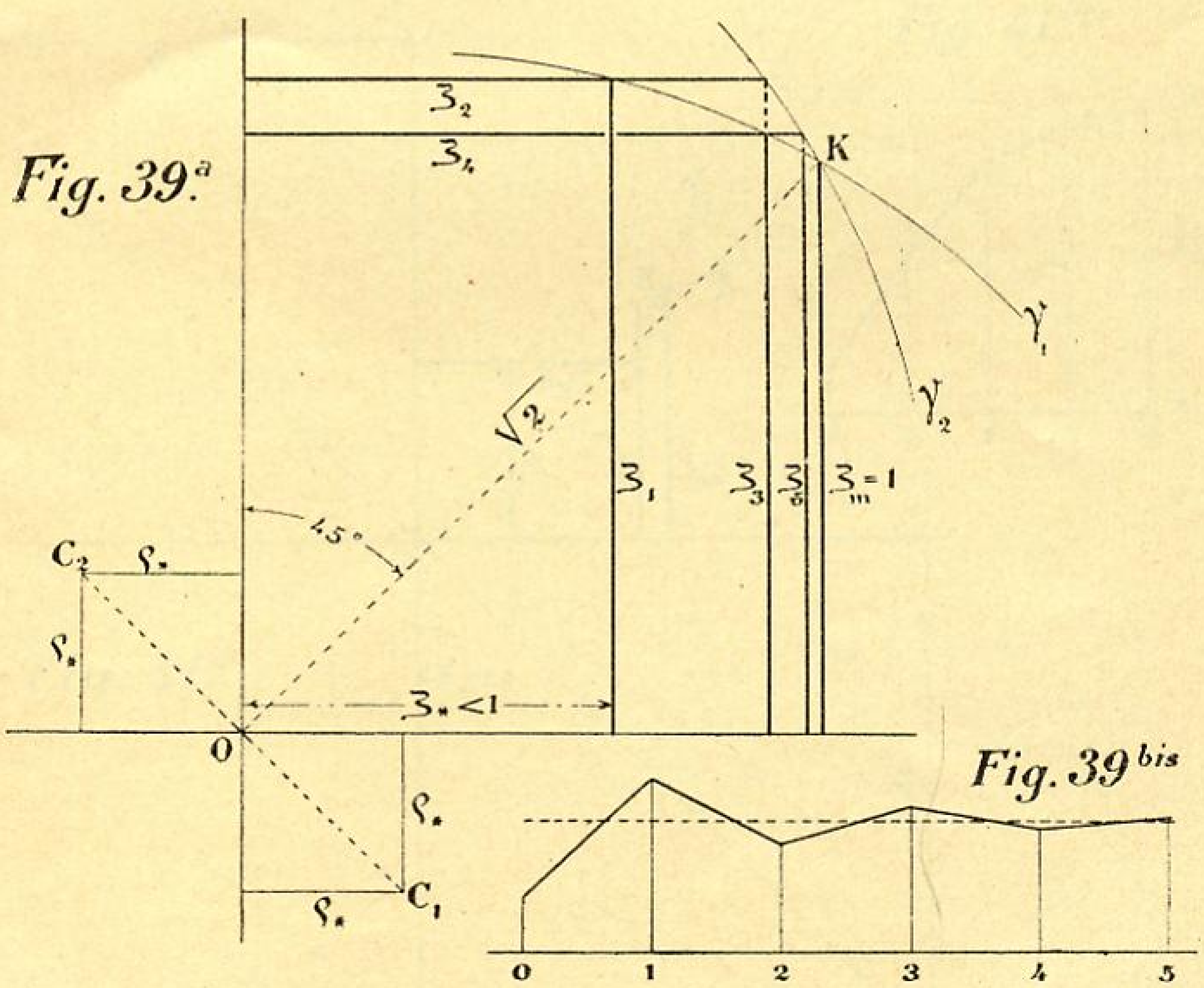


Fig. 39.a

Fig. 39 bis

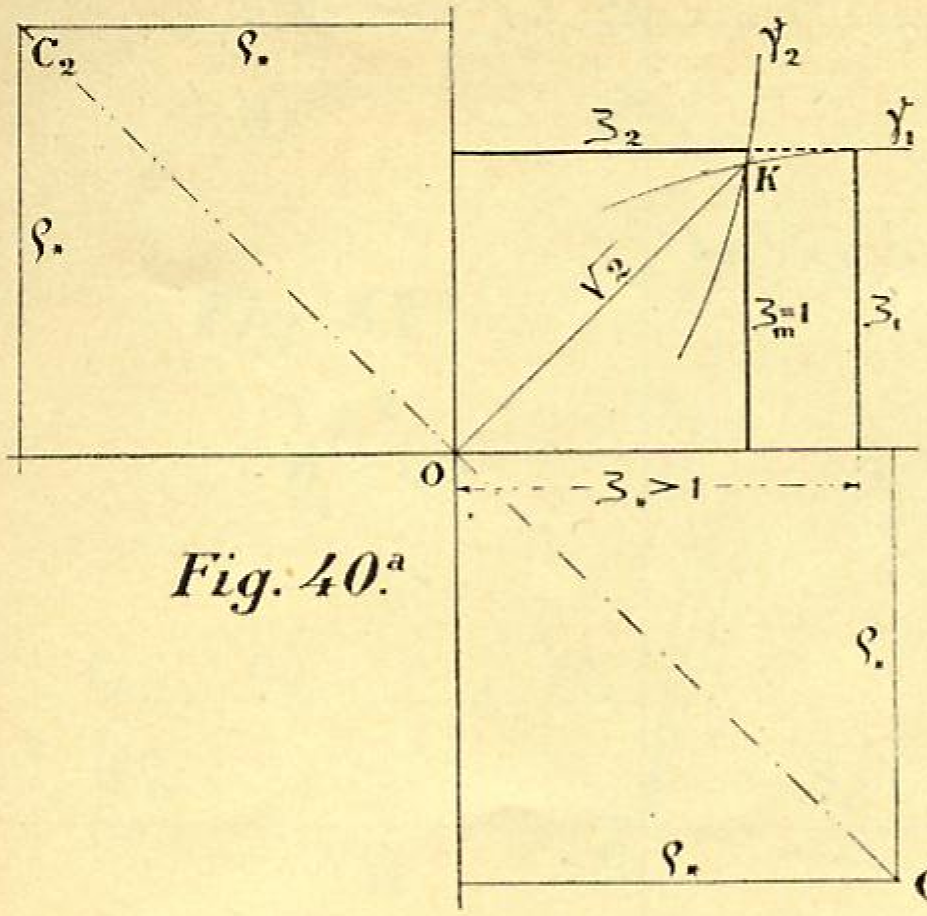


Fig. 40.a

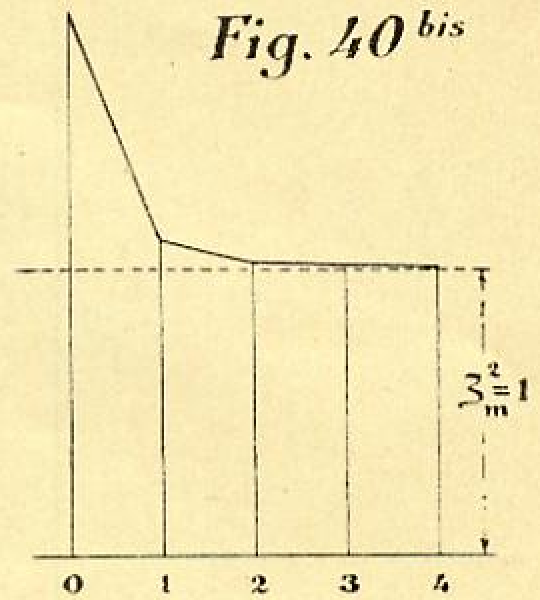


Fig. 40 bis

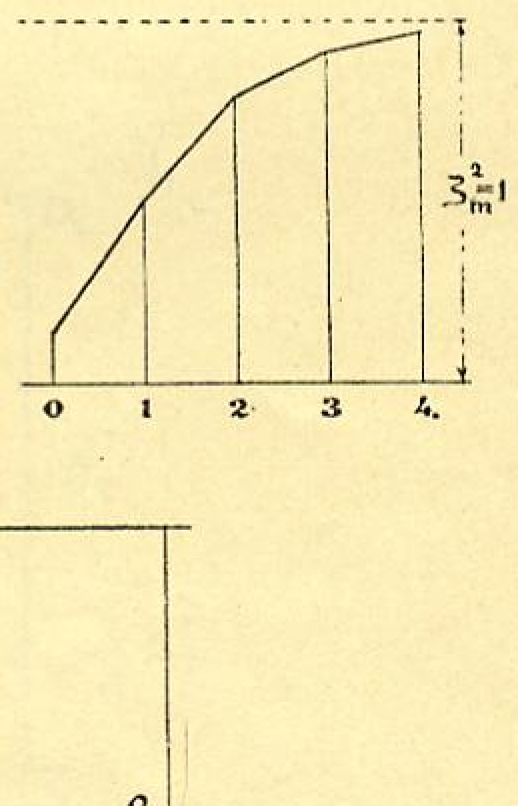


Fig. 41 bis

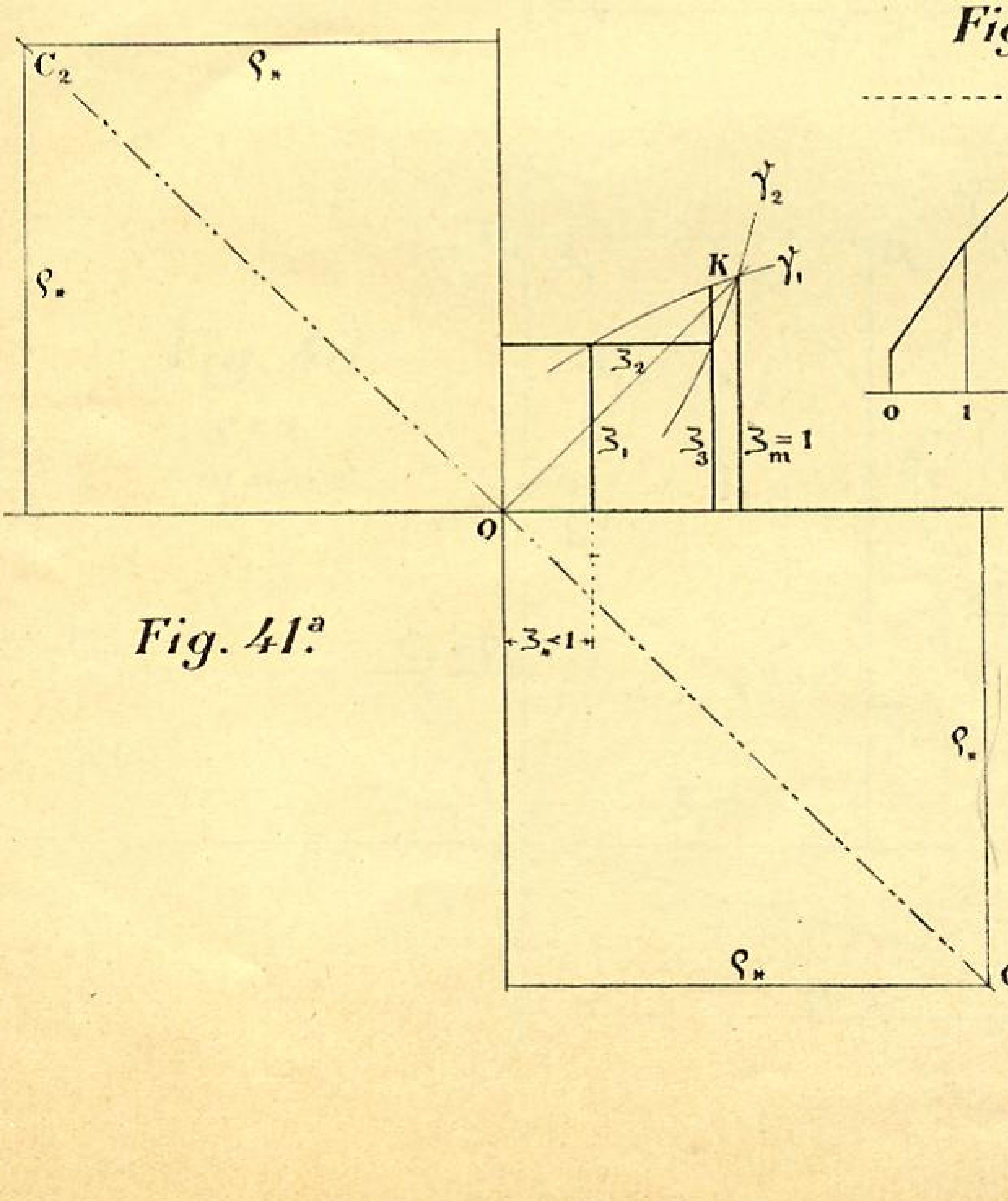


Fig. 41.a

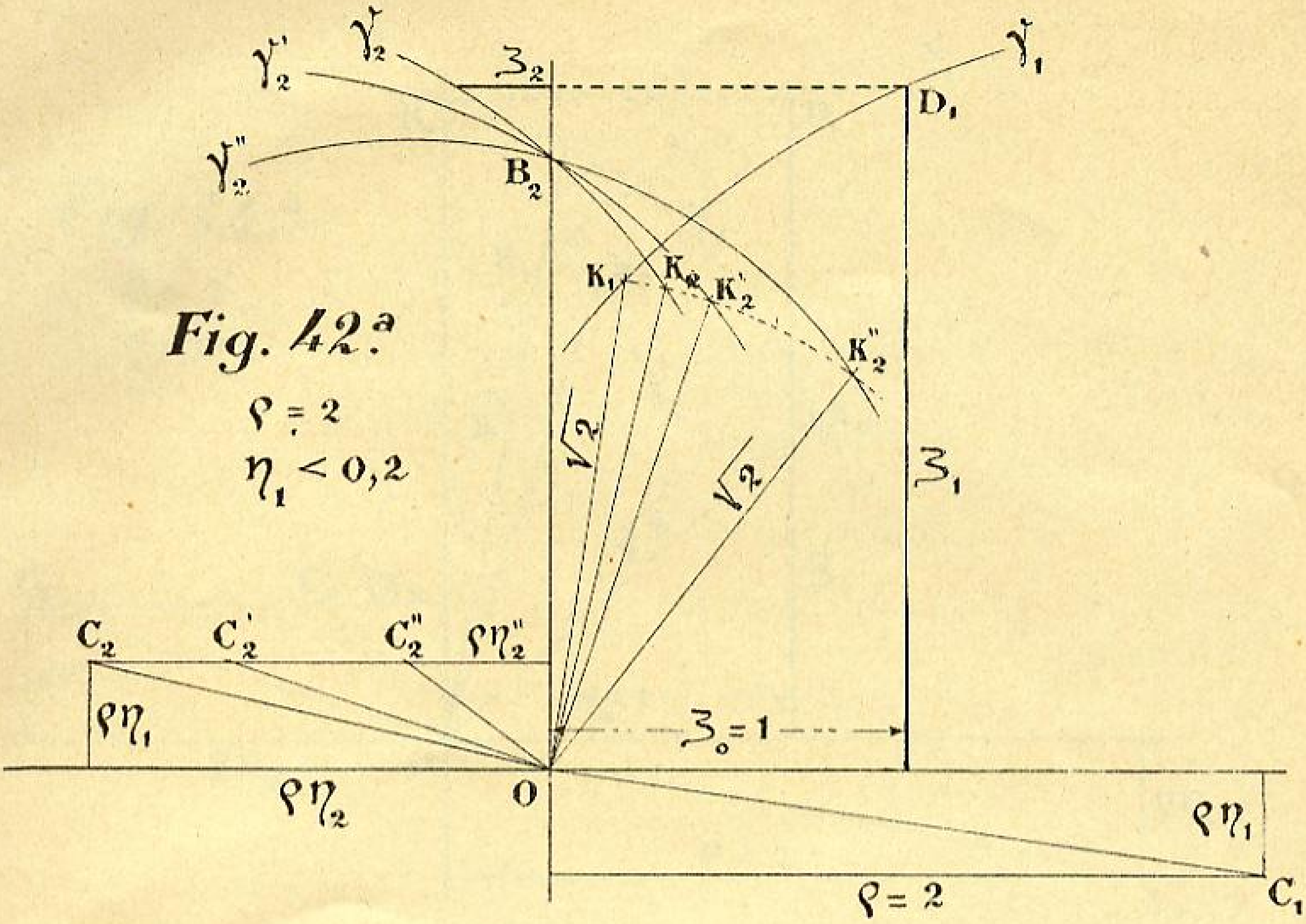


Fig. 42.ª

$\rho = 2$

$\eta_1 < 0,2$

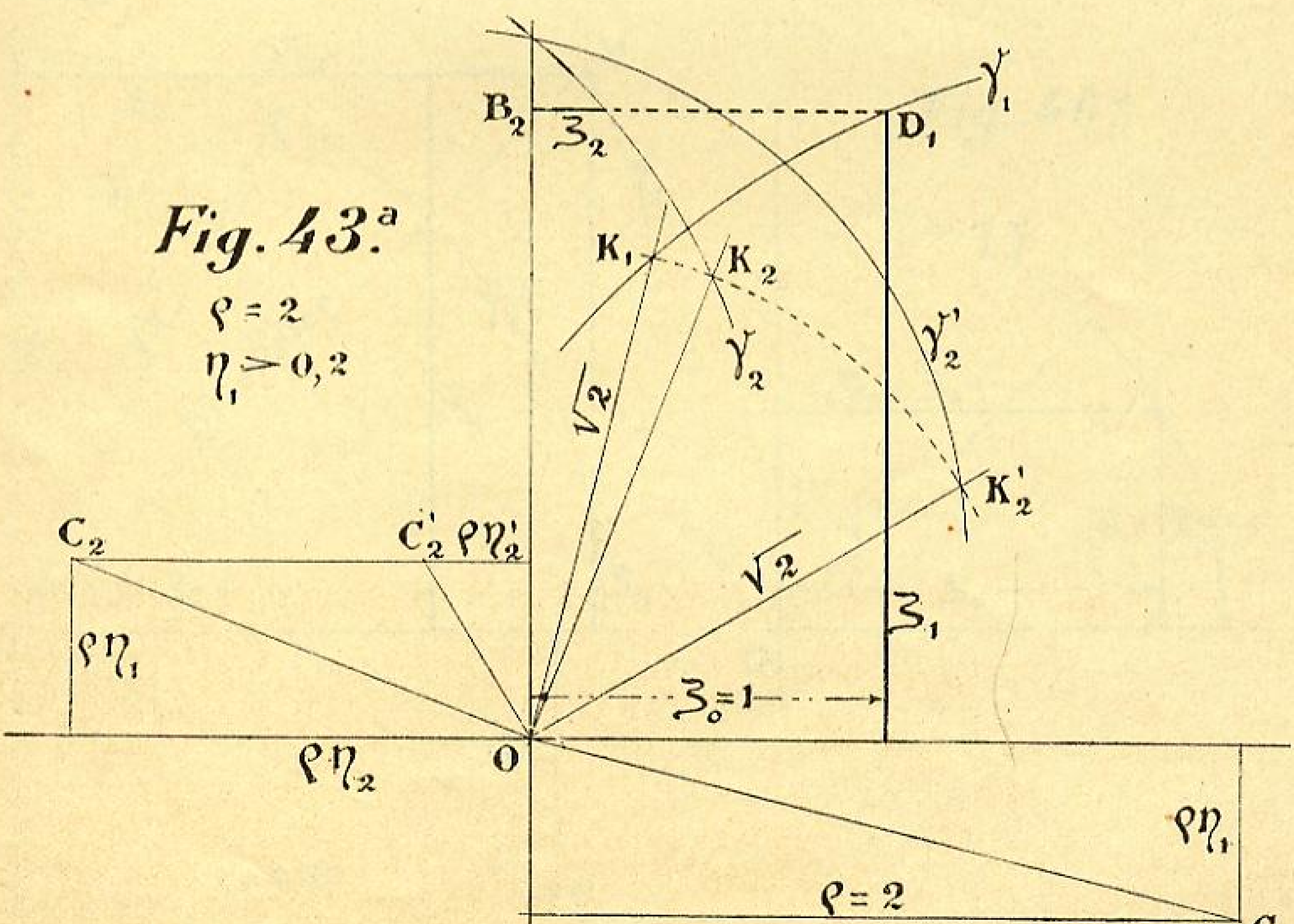


Fig. 43.ª

$\rho = 2$

$\eta_1 > 0,2$

Fig. 44.ª

$\rho = 2$

$\eta_1 = 0,2$

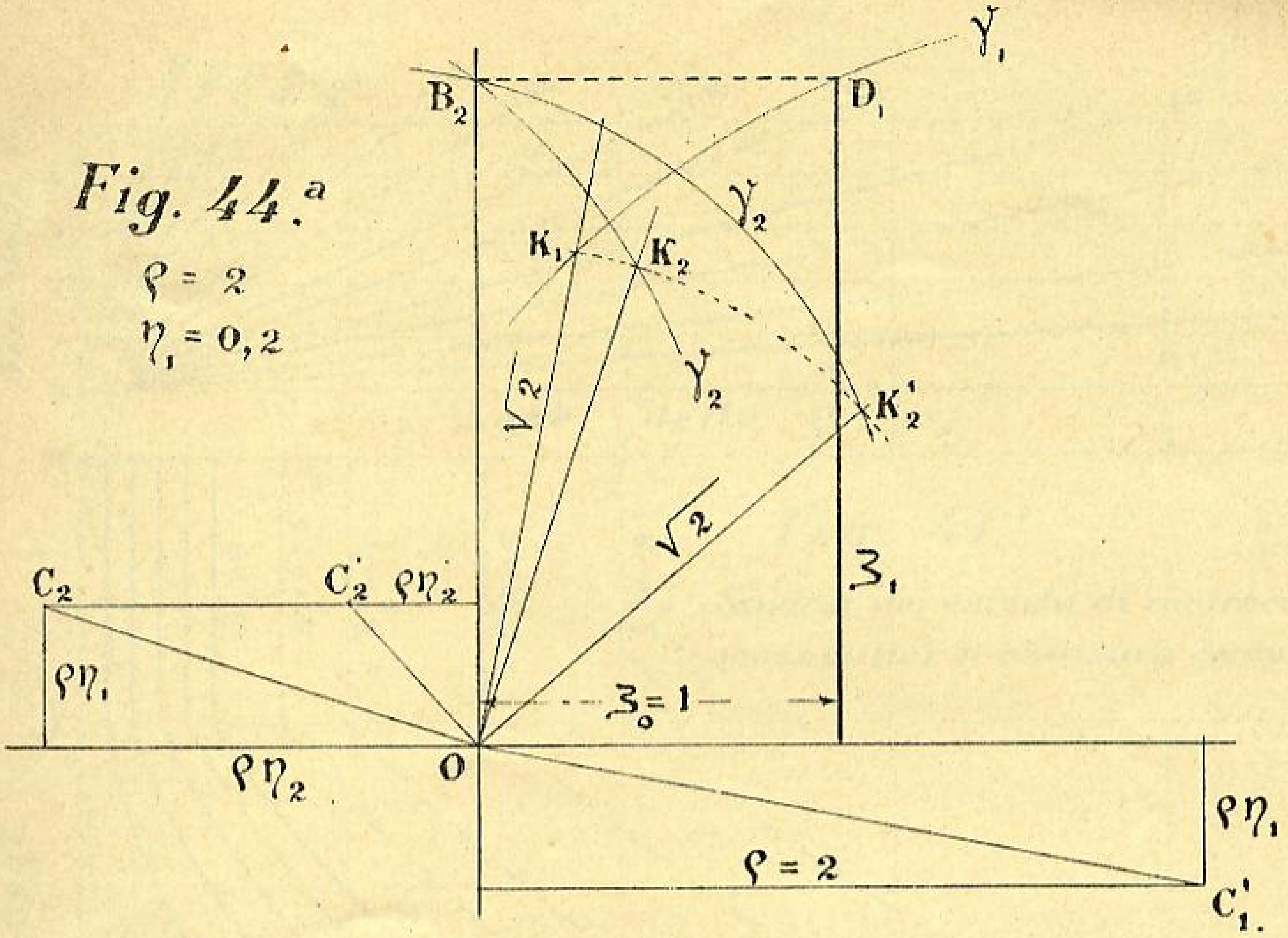


Fig. 45.ª $\rho = 1,3$ $\eta_1 = 0,5$ $\eta_2 = 0$

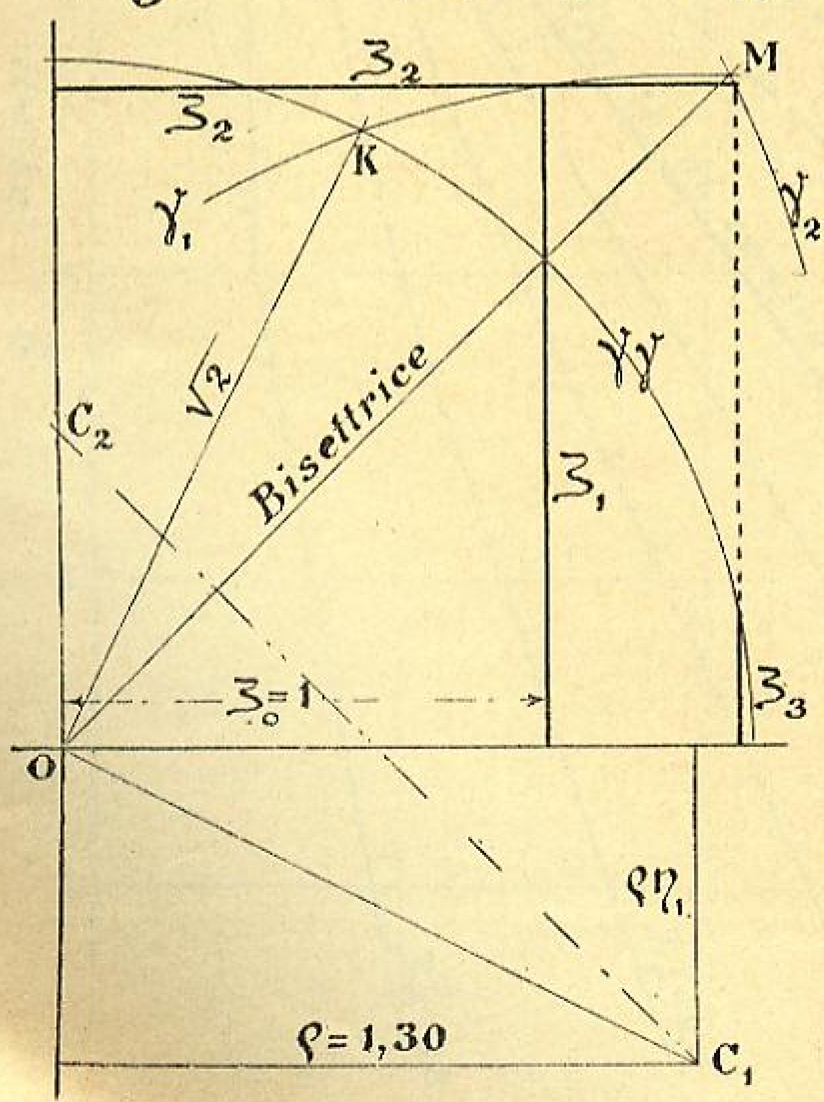
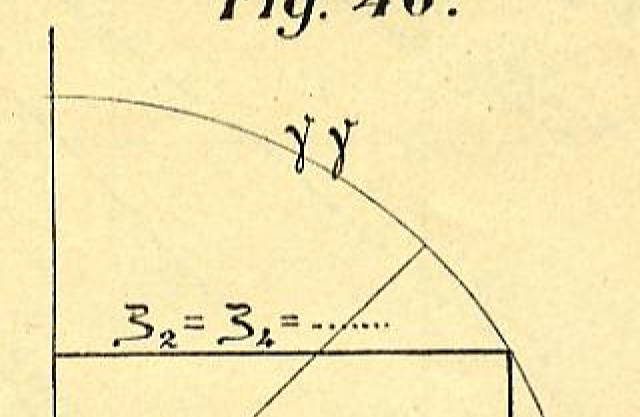


Fig. 46.ª



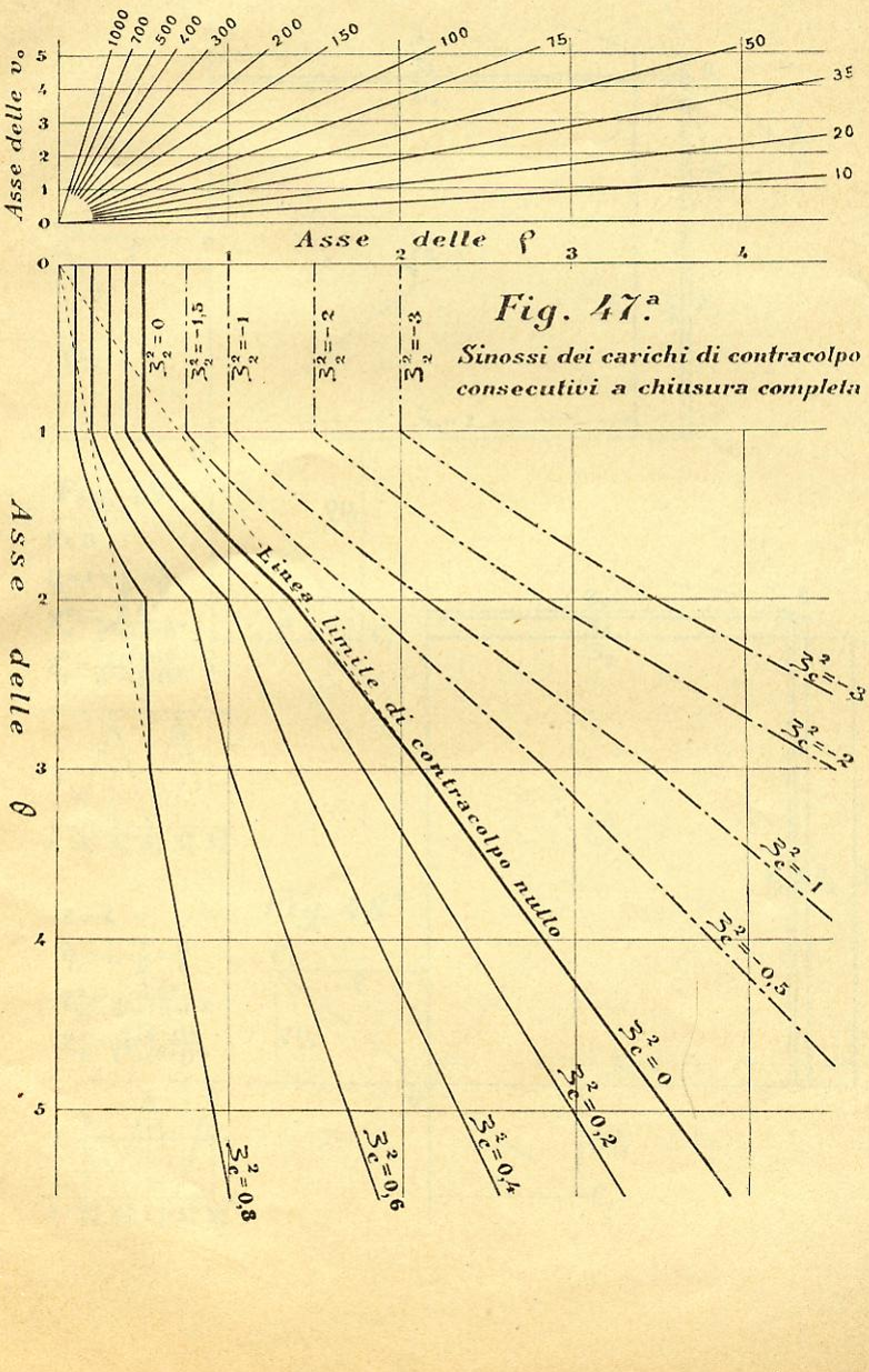


Fig. 47ª

Sinossi dei carichi di contracolpo consecutivi a chiusura completa

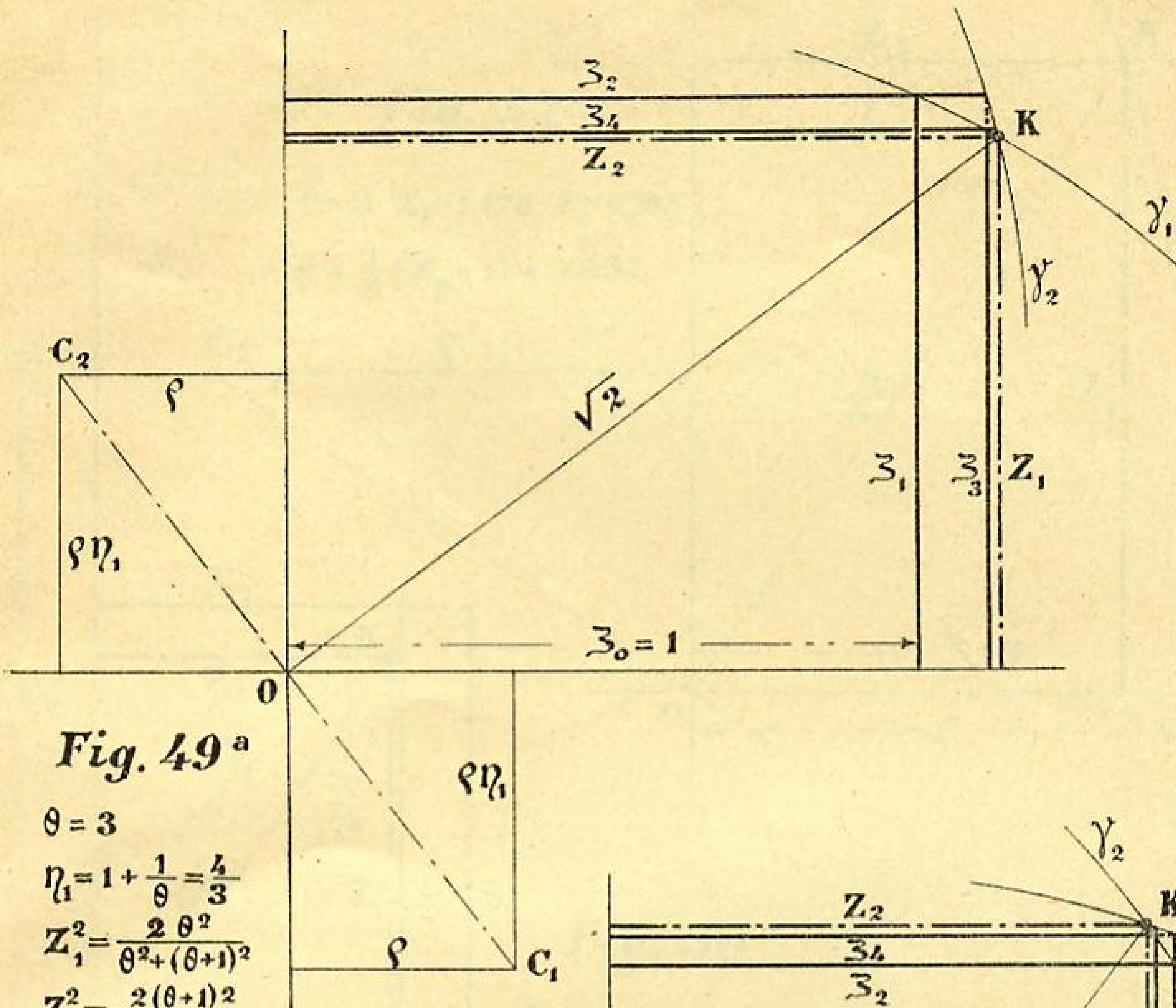


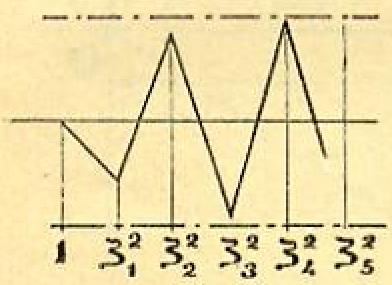
*Fig. 49*ª

$\theta = 3$

$\eta_1 = 1 + \frac{1}{\theta} = \frac{4}{3}$

$Z_1^2 = \frac{2\,\theta^2}{\theta^2 + (\theta+1)^2}$

$Z_2^2 = \frac{2\,(\theta+1)^2}{\theta^2 + (\theta+1)^2}$



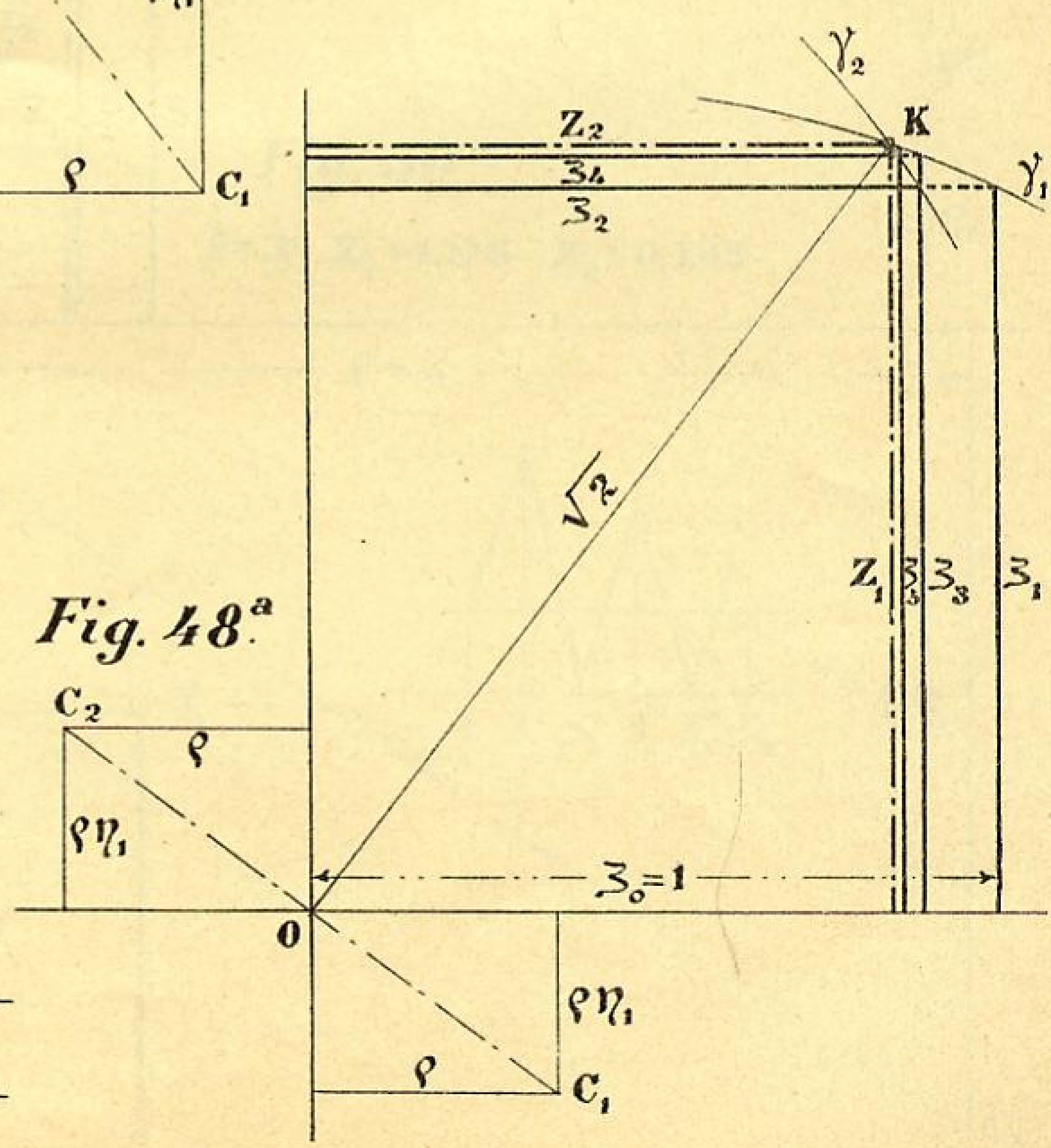


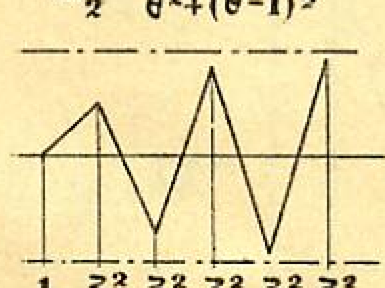
*Fig. 48*ª

$\theta = 4$

$\eta_1 = 1 - \frac{1}{\theta} = \frac{3}{4}$

$Z_1^2 = \frac{2\,\theta^2}{\theta^2 + (\theta-1)^2}$

$Z_2^2 = \frac{2\,(\theta-1)^2}{\theta^2 + (\theta-1)^2}$



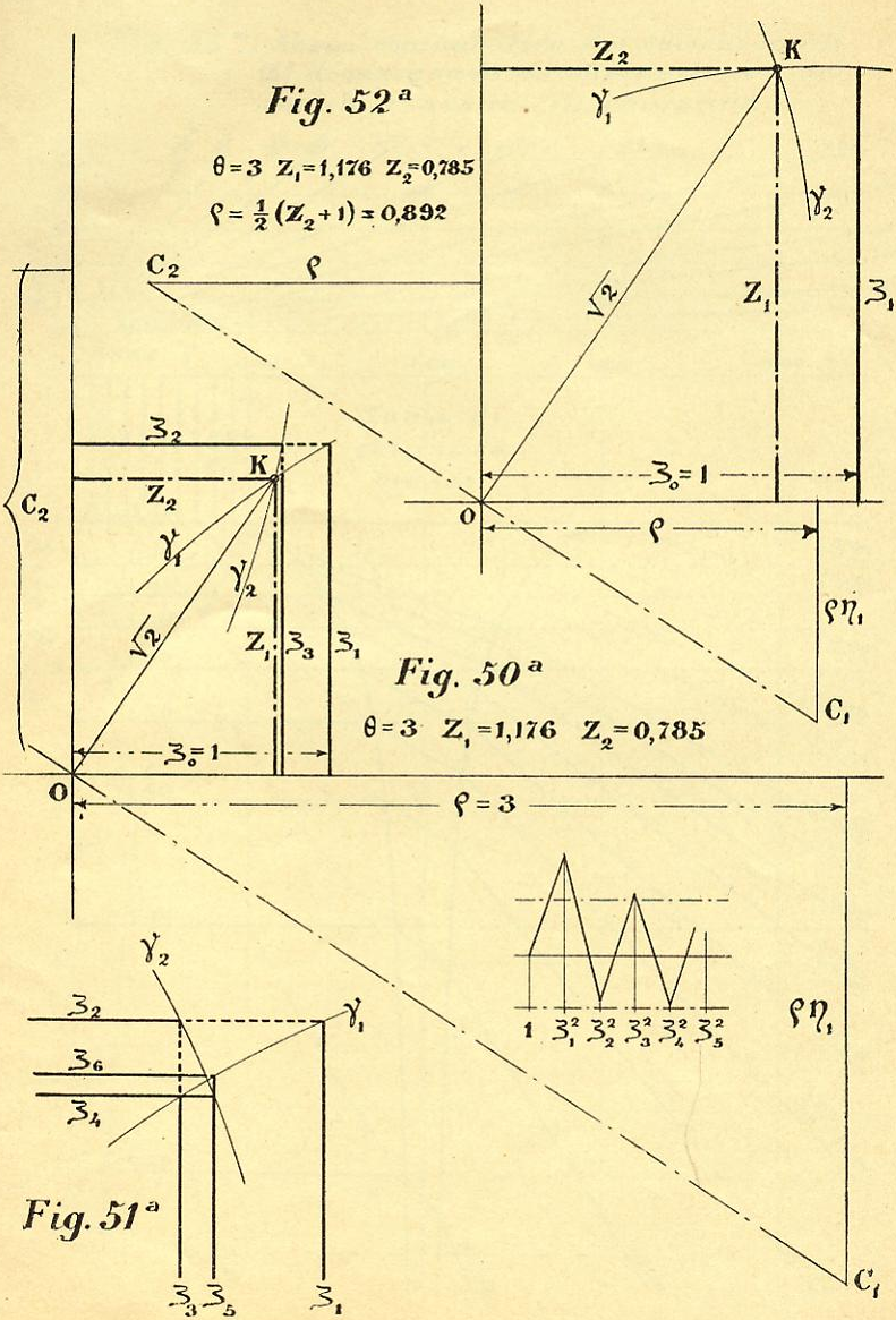


Fig. 52ª

Fig. 50ª

Fig. 51ª

Fig.: 53ª – Abaco comparativo dei carichi-limiti Z_1^2 di risonanza da alternazione e dei carichi massimi in chiusura.

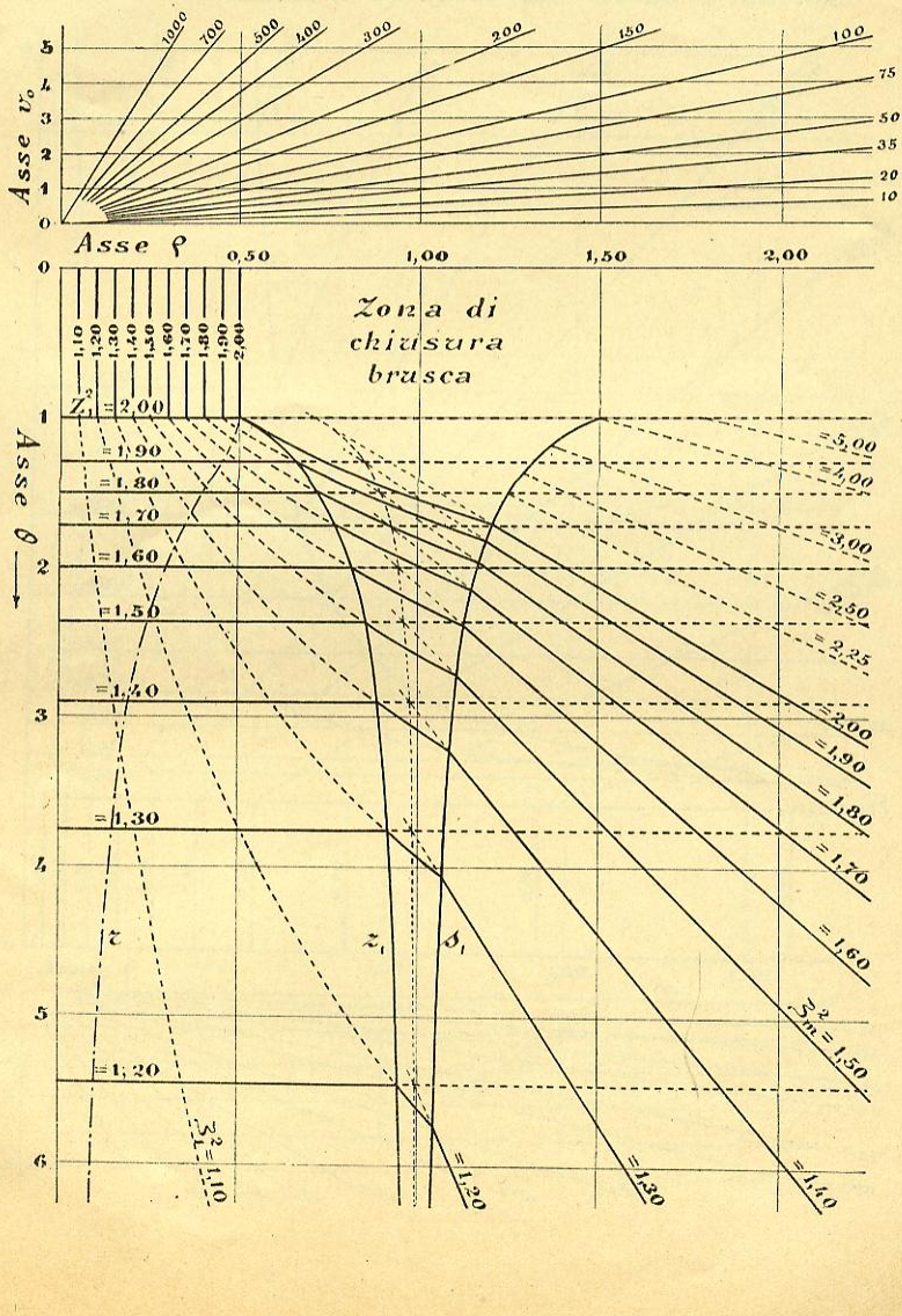


Fig. 54ª – Abaco comparativo dei carichi-limiti Z_2^2 di risonanza da alternazione e dei carichi minimi di apertura per messa in funzione.

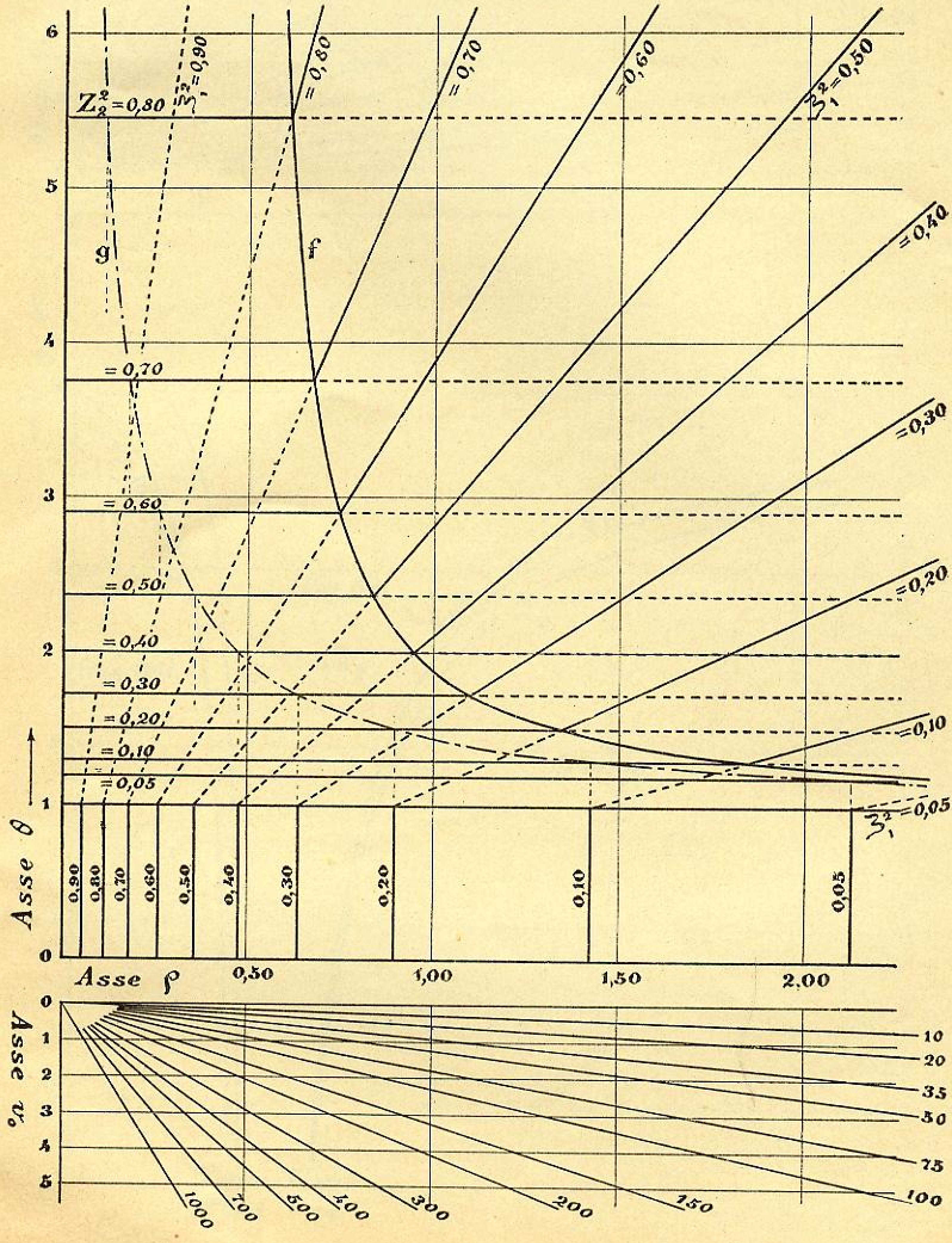


Fig. 55.ª — Sinossi di classifica rispetto alle leggi di risonanza da alternazione.

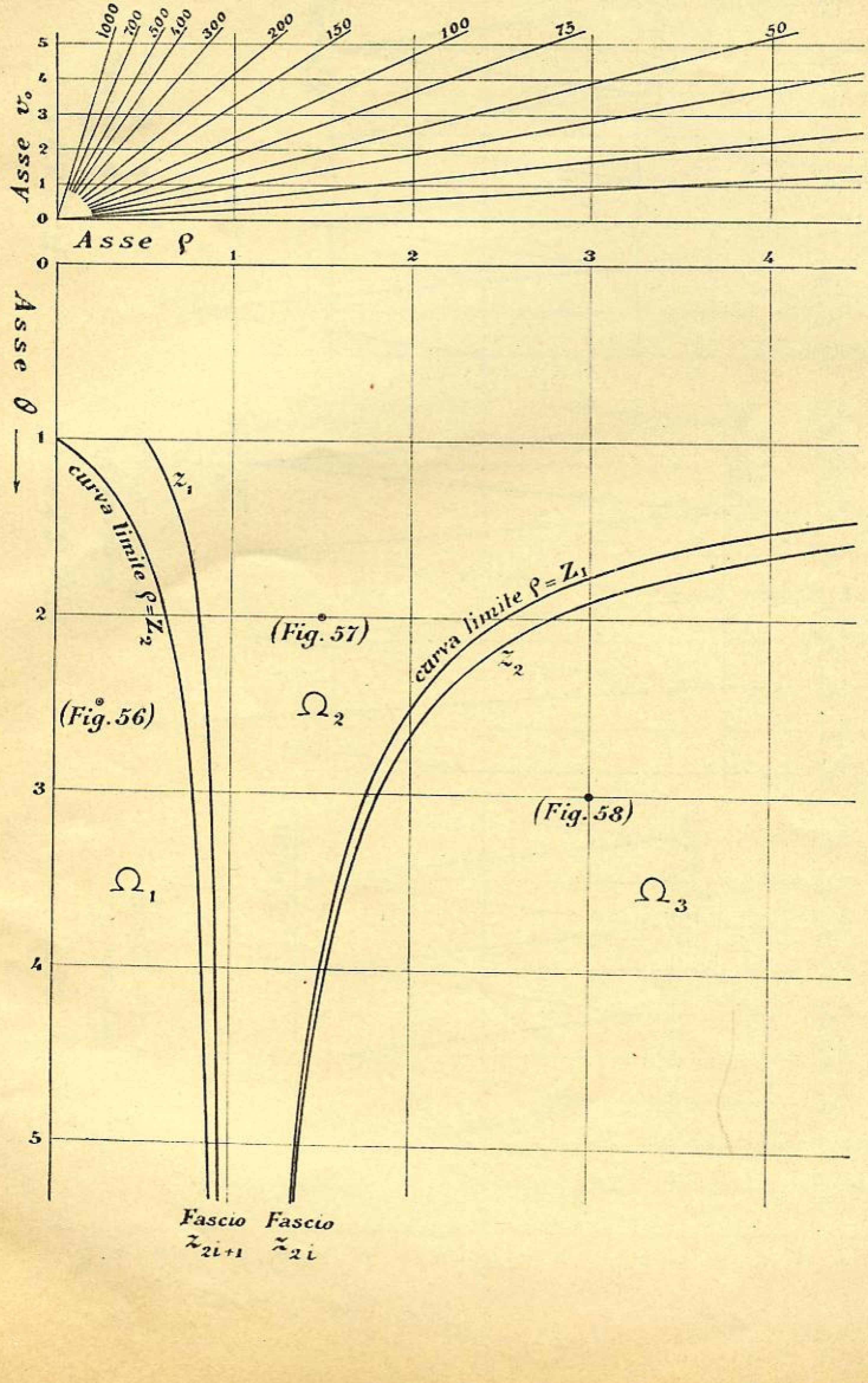


Fig. 56ª — Regione Ω_1, $\theta = 2{,}5$ $\rho = 0{,}25$

Fig. 57ª — Regione Ω_2, $\theta = 2{,}0$ $\rho = 1{,}5$

Fig. 58ª — Regione Ω_3, $\theta = 3$ $\rho = 3$

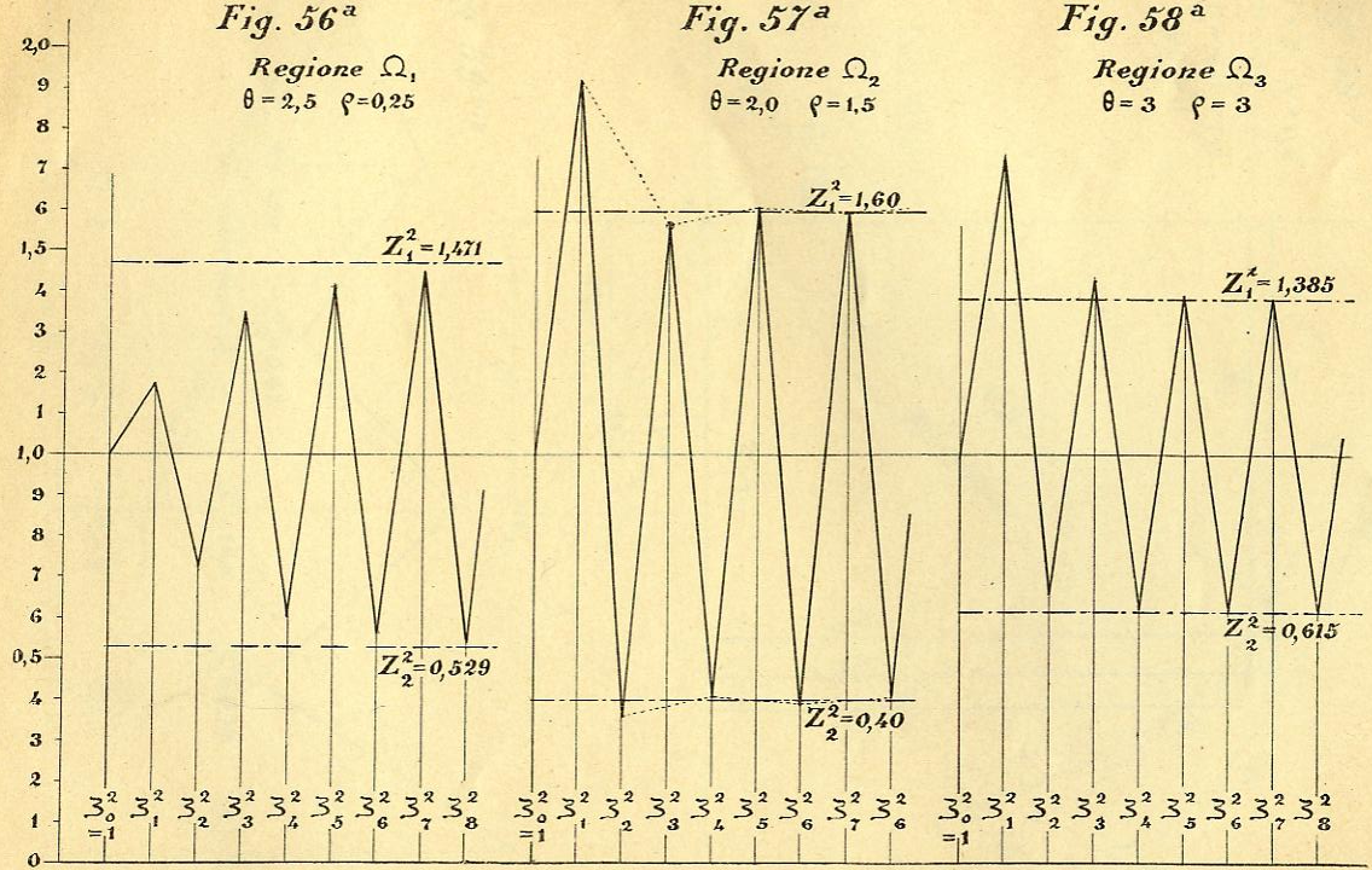


Fig. 59 a

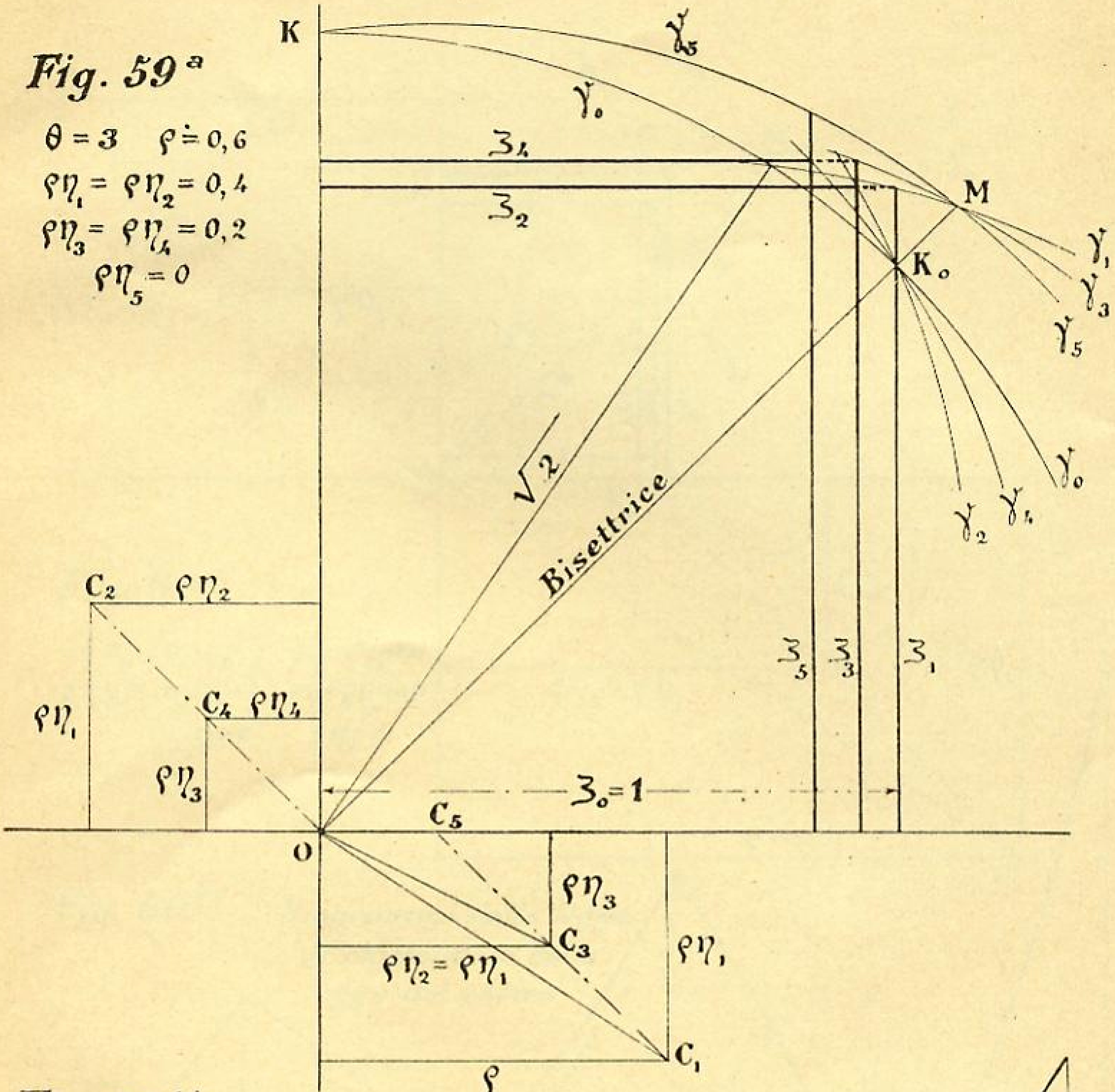
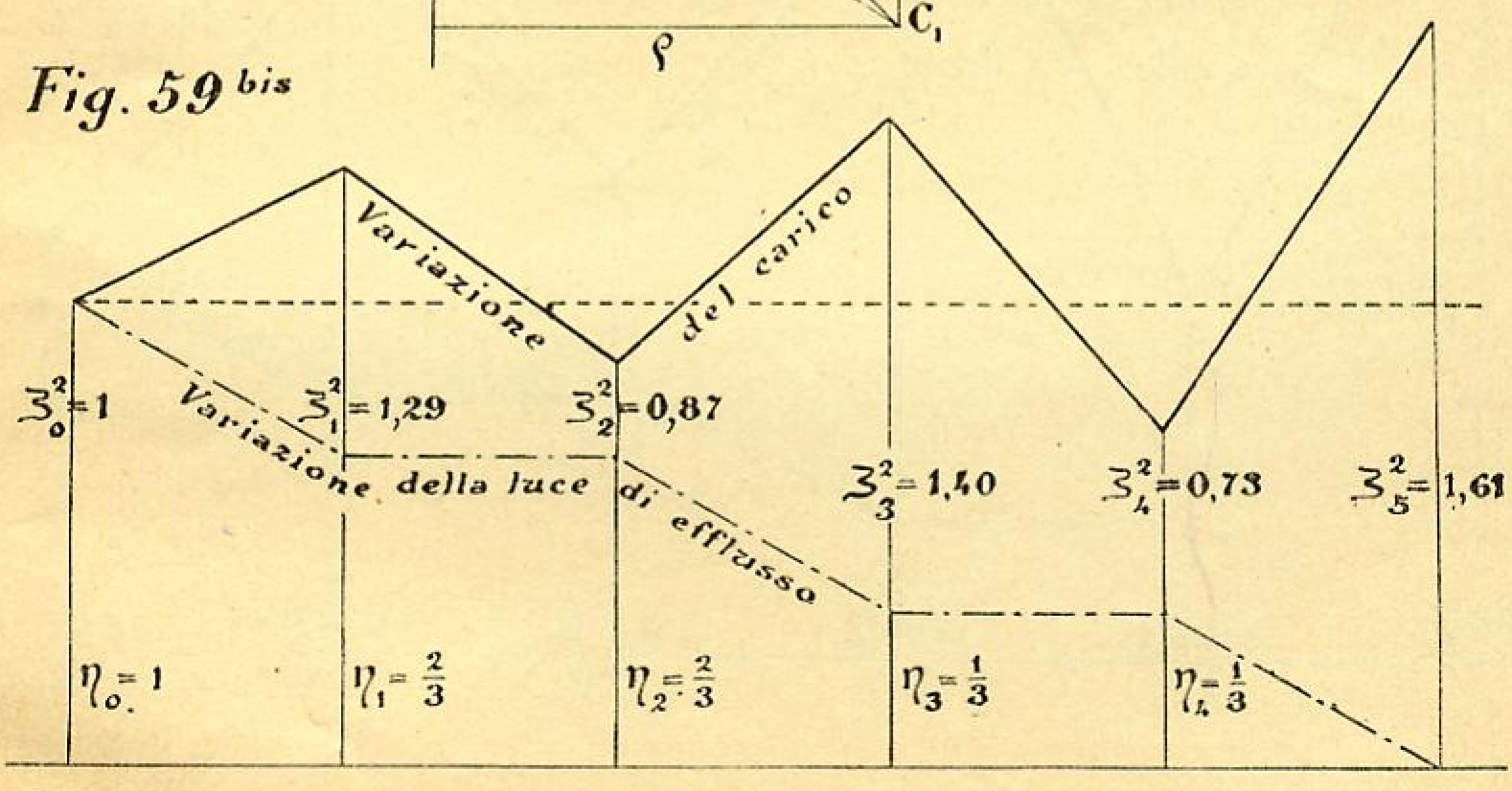


Fig. 59 bis



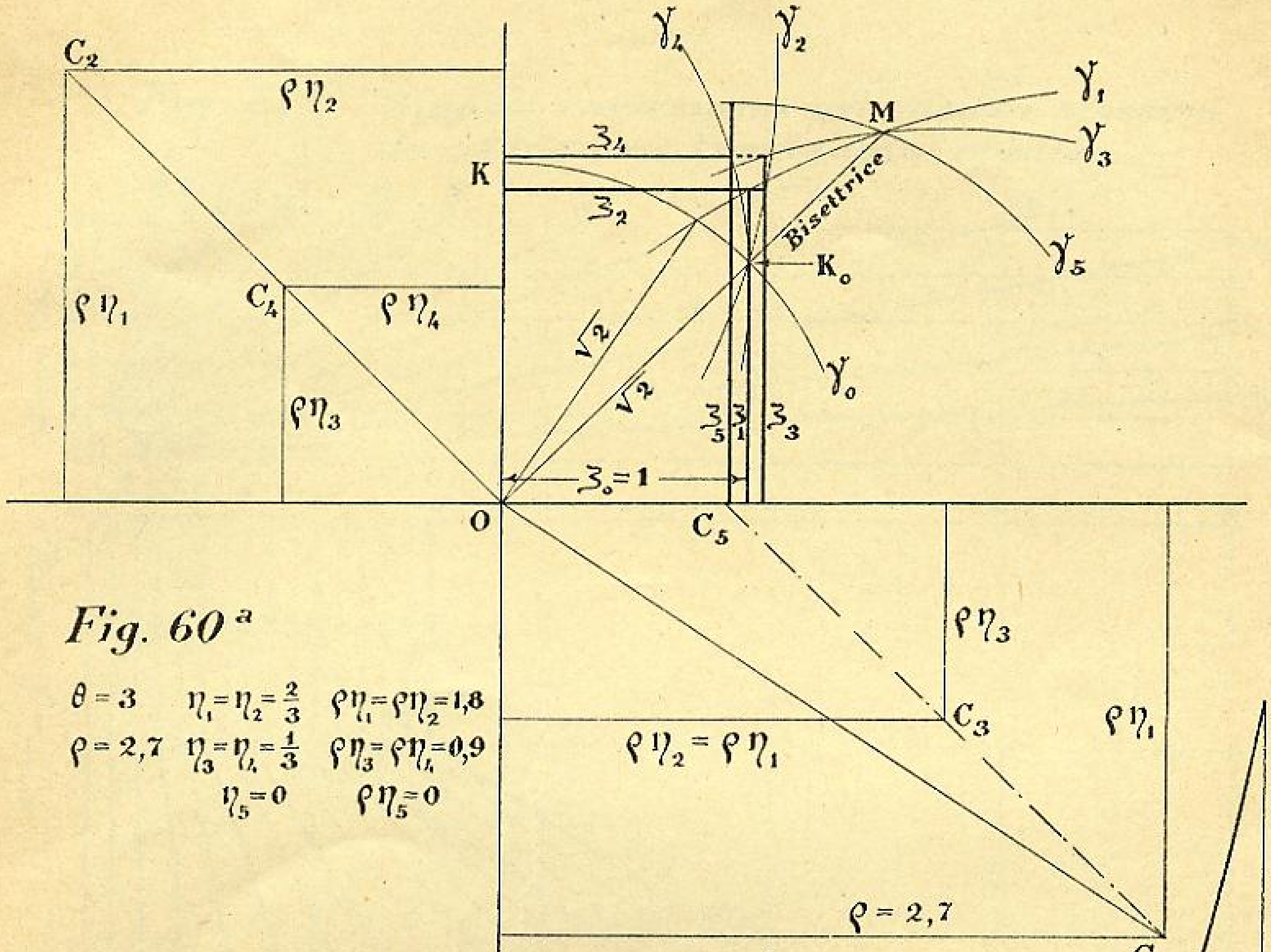


Fig. 60a

$\theta = 3 \quad \eta_1 = \eta_2 = \frac{2}{3} \quad \rho\eta_1 = \rho\eta_2 = 1{,}8$

$\rho = 2{,}7 \quad \eta_3 = \eta_4 = \frac{1}{3} \quad \rho\eta_3 = \rho\eta_4 = 0{,}9$

$\eta_5 = 0 \quad \rho\eta_5 = 0$

Fig. 60bis ***Diagrammi della legge di chiusura e della legge del carico***

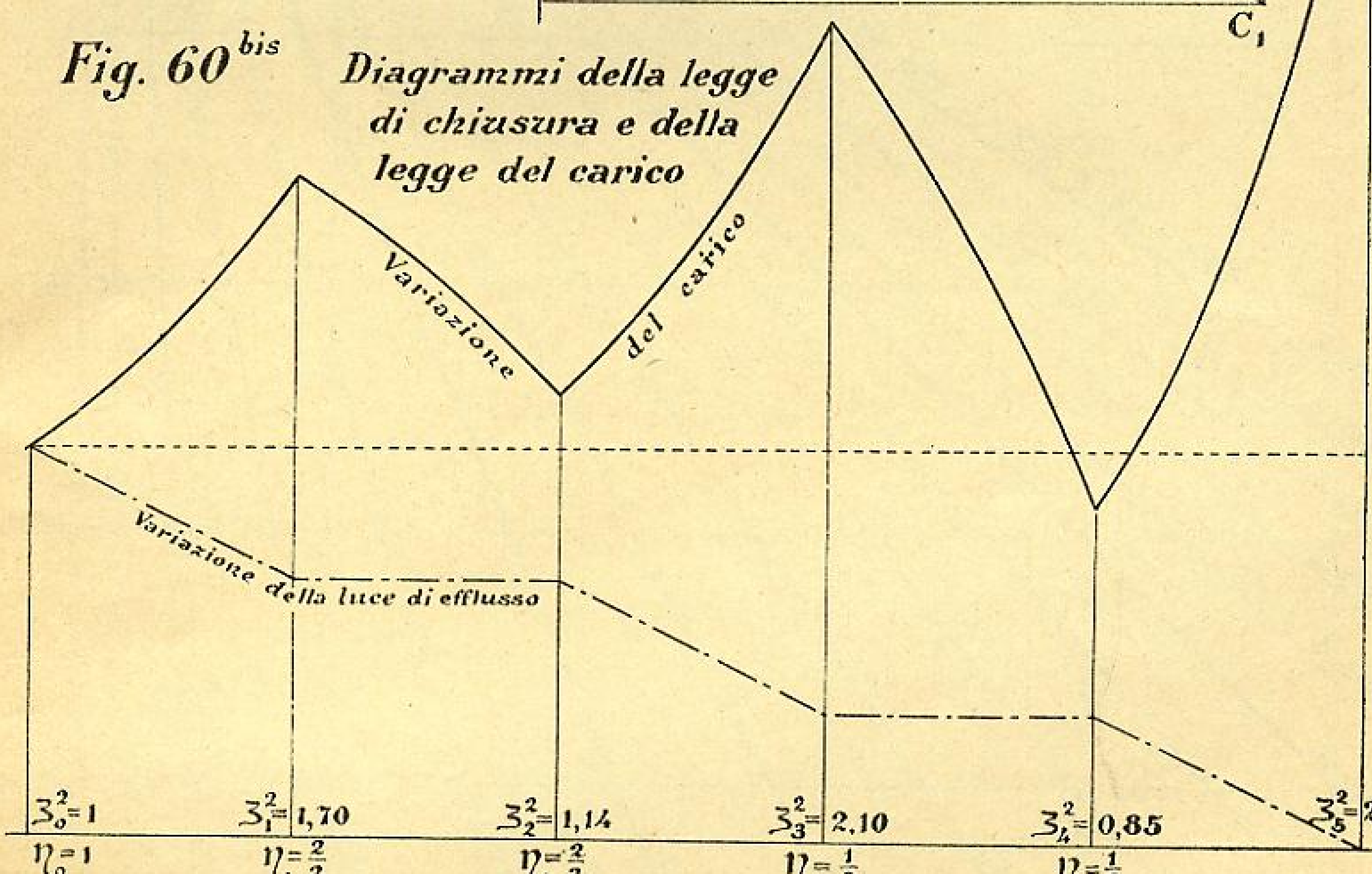


Fig. 61ª *Sinossi cartesiana pei carichi massimi da chiusura frazionata ritmica.*

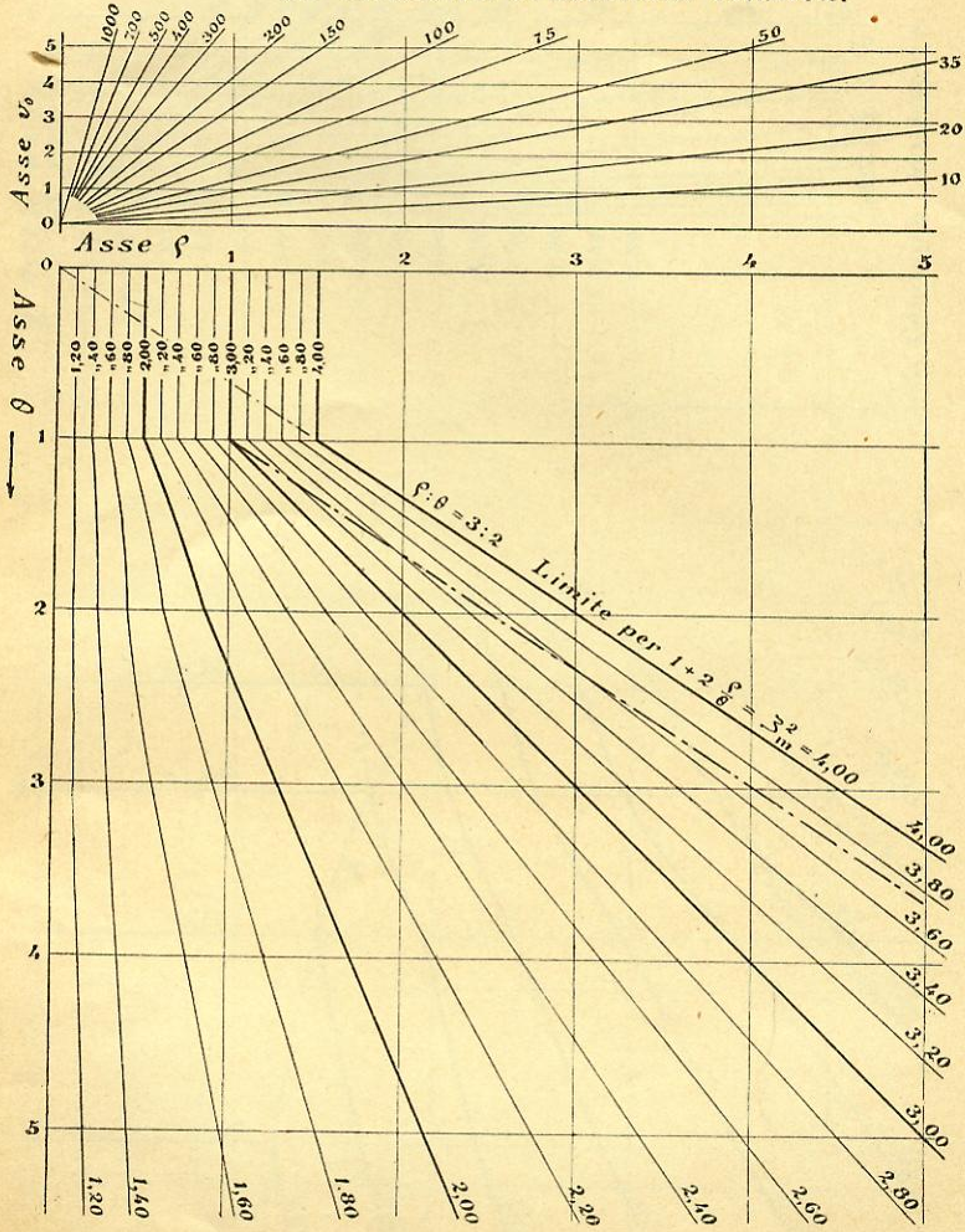
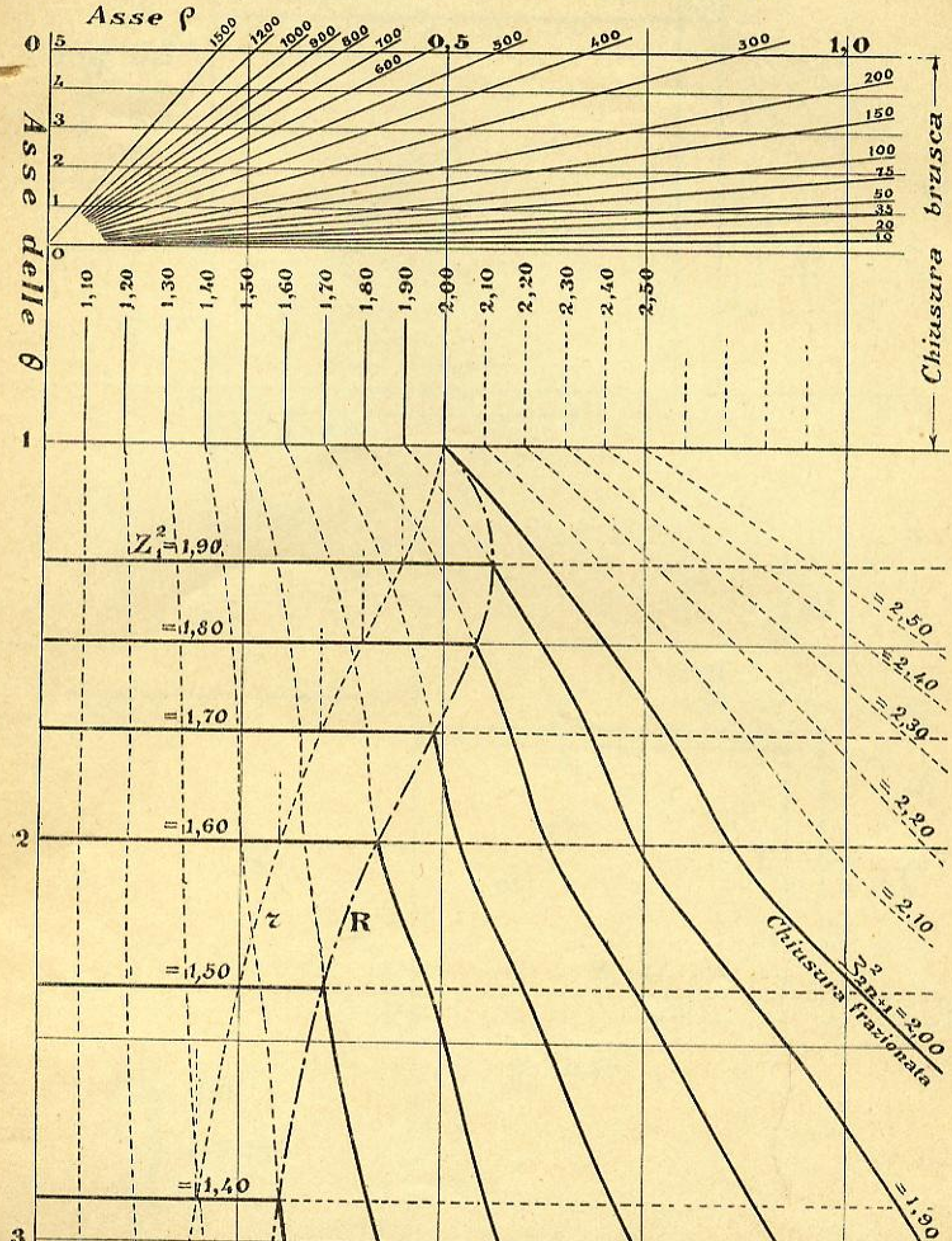


Fig. 62ª — Abaco di confronto fra i massimi di risonanza da alternazione e da chiusura frazionata ritmica.



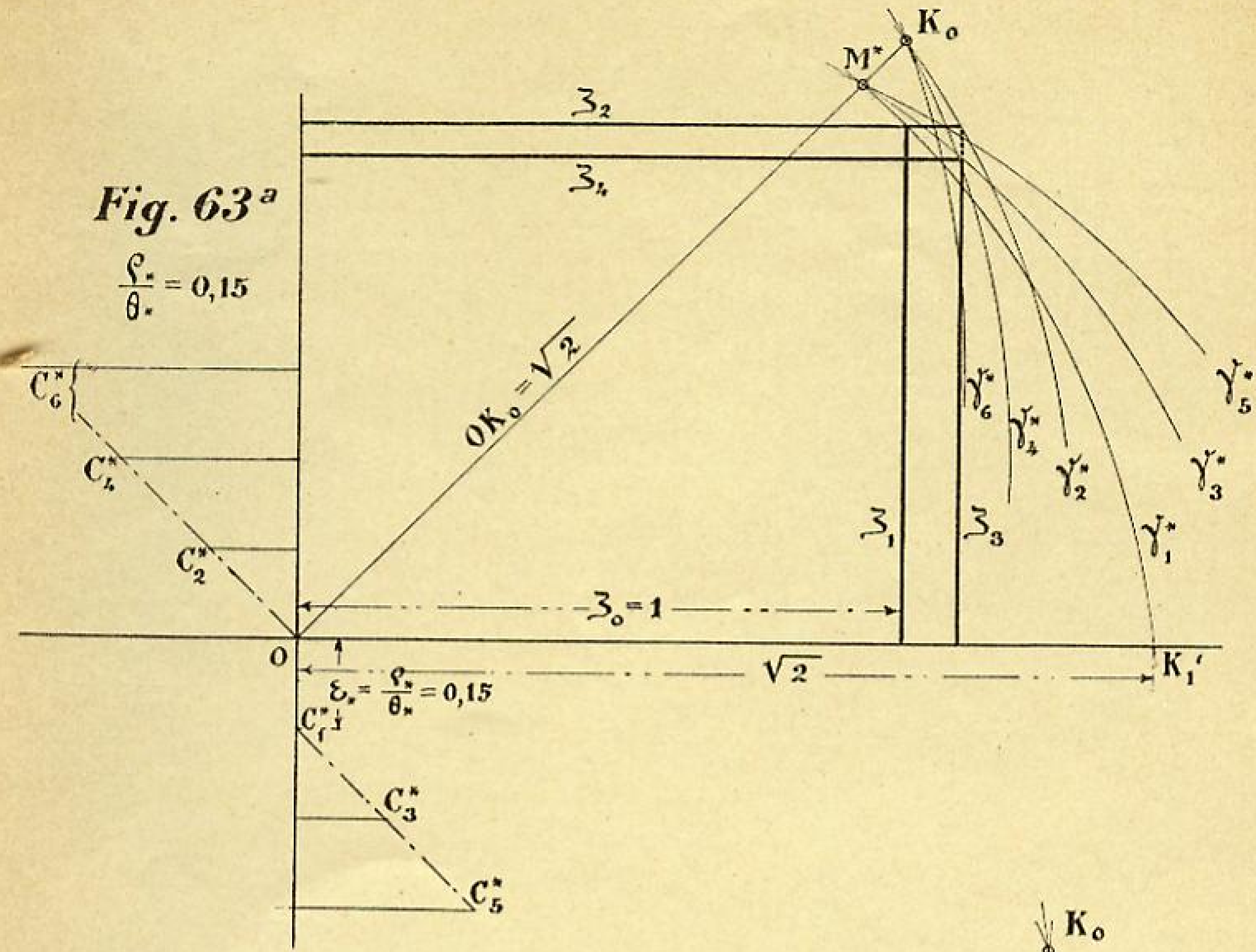


Fig. 63[a]

$\frac{\rho_*}{\theta_*} = 0{,}15$

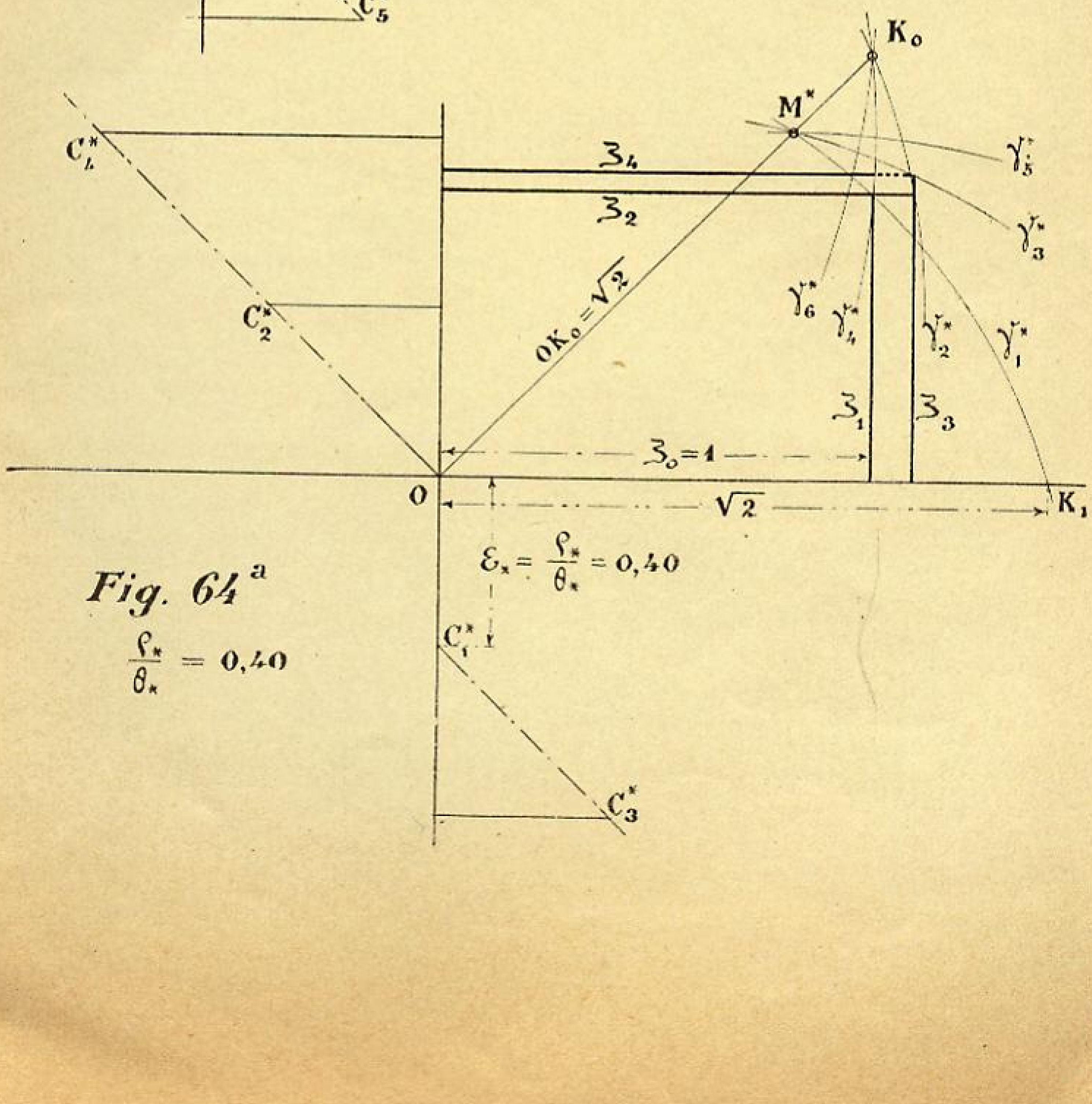


Fig. 64[a]

$\frac{\rho_*}{\theta_*} = 0{,}40$