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WITH DISCUSSION BY

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Synopsis.

This paper shows how the rise of pressure caused by the gradual closing of turbine gates may be determined from Professor Joukovsky's theory of maximum water-hammer. A solution of the problem by the trial-and-error method of arithmetic integration is first given, and then formulas are derived that cover any governor time and any relation between governor stroke and gate movement. Two factors which affect the rise of pressure are taken into consideration: (1) the elasticity of water and of the walls of the penstock; and (2) the effect of the net head on the phenomena which occur during the closing of the gates. Previous writers have submitted formulas which neglect one or other of these two factors, with the result that they do not give correct results in all cases. The limitations of such formulas are determined by the solution of the problem herein submitted.

Introduction.

The purpose of this paper is to show how the excess pressure in penstocks caused by the gradual closing of turbine gates may be determined from Professor Joukovsky's theory of water-hammer. It will be assumed that the theory of pressure waves, their amplitudes, and speeds of propagation, as formulated by him and proved by his experiments, may be accepted as correct.

At the risk of wearying the reader who is familiar with Joukovsky's work, it is necessary, in connection with what is to follow, to summarize as briefly as possible the principles demonstrated a number of years ago by this distinguished Russian. For a partial translation of his work, the reader is referred to Miss O. Simin's paper entitled "Water Hammer"* which should be examined carefully by every student of this subject.

Joukovsky's experiments, made in 1898 at Moscow, were confined to the instantaneous stopping of the flow of water in long pipes. By his experiments he was able to prove the soundness of his analytical determination of the maximum rise of pressure that would occur when the flow of water in a pipe was suddenly arrested. The casual thinker at first would imagine that—as force is equal to the product of mass by acceleration—an infinite pressure would be produced in a pipe if the water flowing in it were stopped instantaneously. On second thought, he would realize that neither the water column nor the walls of the pipe are rigid, and, therefore, the pressure caused by the shock of stopping the flow suddenly is relieved by the slight compression of the water and the expansion of the walls of the pipe. It was the effect of these two factors that was determined by Professor Joukovsky. He showed that the shock pressure is transmitted along the column of water in the pipe in waves similar to sound waves; and that the shock pressure is proportional to the destroyed velocity of flow and to the speed of propagation of the pressure waves. This speed depends on the compressibility of water, on the elasticity of the materials of the pipe, and on the ratio of the thickness of the walls of the pipe to its diameter. In other words, if the speed of the pressure wave is known, the maximum pressure produced (called water-hammer) by instantane-

* *Proceedings*, Am. Water Works Assoc., 1904, p. 341.

ously stopping water flowing in a pipe at any velocity may be calculated.

Joukovsky's formula for water-hammer is:

$$h = \frac{aV}{g} \qquad (1)$$

where h = excess pressure, in feet;

V = velocity of flow in the pipe, in feet per second;

g = gravitational unit, in feet per second per second;

and a = velocity of the pressure wave, in feet per second, which is determined by the formula:

$$a = \frac{12}{\sqrt{\frac{W}{g}\left\{\frac{1}{k} + \frac{d}{Ee}\right\}}} \qquad (2)$$

where W = weight of a cubic foot of water, in pounds;

g = gravitational unit, in feet per second per second;

k = voluminal modulus of water, in pounds per square inch;

d = diameter of pipe, in inches;

e = thickness of pipe walls, in inches;

E = modulus of elasticity of material of pipe walls, in pounds per square inch.

Joukovsky showed also that the shock pressure is transmitted along the pipe with constant intensity and "at constant velocity, which seems to be independent of the intensity of the shock."

"The speed of propagation of the pressure wave remains the same, whether the shock is caused by arresting the flow of a column of water moving in a pipe, or by suddenly changing the pressure in the column of water (flowing or standing) in any part and by any other means."

"If the water column continues flowing, such flow exerts no noticeable influence upon the shock pressure. In a pipe from which water is flowing, the pressure wave is reflected from the open end of the pipe, in the same way as from a reservoir with constant pressure."

"The phenomenon of periodical vibration of the shock pressure is completely explained by the reflection of the pressure wave from the ends of the pipe, *i. e.*, from the gate and from the origin [of the pipe]."*

Pressure waves, after traveling up the penstock to the origin or point of relief and back to the gate, are reflected and transmitted again

* Quotations are from "Water Hammer," by Miss O. Simin.

over the same course, as waves of rarefaction or sub-normal pressure. In this manner pressure waves, alternately super-normal and sub-normal, travel up and down the penstock until damped out by friction. During the time taken by the wave to traverse its course from gate to origin the pressure at the gate remains at its full value, either super-normal or sub-normal, as the case may be.

Briefly stated, then, the premises are: that when water flowing in a pipe is suddenly arrested, certain pressure waves, the characteristics of which are known, are produced and propagated along the pipe at constant speed and constant magnitude, and that the speed and magnitude of these waves may be calculated for any given conditions.

Fundamental Equations.

It is the writer's intention to apply herein this theory of pressure waves to the phenomena which occur when the gates of a turbine at the end of a penstock are gradually closed. The damping effect of friction on the pressure waves will be neglected during the time of closure. The variable velocity of the pressure wave due to the difference in density of the water at the top and bottom of the penstock will also be neglected. For the sake of simplicity, it will be assumed that the gate opening is closed uniformly from full open to shut by a governor which moves the gates at uniform velocity from the beginning of its stroke to the end, and that the area of gate opening is directly proportional to the amount of gate movement. It may be stated here, parenthetically, that the resulting formulas may be modified, easily, to suit any method of gate closure, whether the speed of closing and the relation of governor movement to area of gate opening is uniform or variable.

When the gates of a turbine are closed gradually the velocity of the water in the penstock is reduced to zero and the pressure in the penstock rises. It is frequently assumed that the reduction in velocity takes place uniformly, but the rise of pressure, which commences immediately after the gates begin to move, increases the velocity of discharge through the gate opening, and, during the early part of the gate movement, tends to diminish the rate at which the flow of the water is retarded. This variable rate of retardation, during the time the gates are being closed, has an important bearing on the resulting rise of pressure, and it is necessary to take it into consideration by determining the relation between the velocity of flow and the pressure in the penstock. This relation may be expressed by the equation:

$$V_0 = B_0 \sqrt{H_0} \quad \ldots\ldots (3)$$

where V_0 = initial velocity, in feet per second, of the water in the penstock before shut down;

H_0 = normal net head, in feet;

and B_0 = a number representing the gate opening. The value of B_0 is best determined from the known value of V_0 and H_0, but, in fact, B_0 is the ratio of the area of the gate opening to the area of the penstock multiplied by $\sqrt{2g}$ times the coefficient of discharge of the gate opening.

At any time during the closing of the gates the relation between V, B, and H would be expressed by the general formula, $V = B\sqrt{H}$. If the gates are closed in the time, T, and t is the time from the beginning of the stroke to any time before the end of the stroke, the value of B_t (that is, B at the end of the time, t) for uniform closing, would be $\left(1 - \frac{t}{T}\right) B_0$. During the time, t, the pressure in the penstock has risen an amount, h_t, so that the net head, H_t (that is, H at the end of the time, t) would be equal to $H_0 + h_t$. Therefore, the expression for the value of V_t (that is V at the end of the time, t) is:

$$V_t = \left(1 - \frac{t}{T}\right) B_0 \sqrt{H_0 + h_t} \quad \ldots\ldots (4)$$

Calculation by Arithmetic Integration.

Before proceeding with the analytical determination of h_t, it will perhaps make the work clearer to show first, by a numerical example, how h_t may be otained by the trial-and-error method of arithmetic integration. Assume that the gate, instead of being moved in a continuous uniform manner, is closed by a series of small instantaneous movements with a slight pause between each movement. Each little movement of the gate would destroy instantaneously a small part, ΔV, of the velocity, V_0, and, since this part of the velocity is destroyed instantaneously, the rise of pressure, according to Joukovsky, would be $h = \frac{a \Delta V}{g}$. When the first instantaneous movement has taken place, let a pause of time, t, elapse before the next movement. Then $\Delta V = V_0 - V_t$, and $h_t = \frac{a(V_0 - V_t)}{g}$.

By these equations and Equation (4) a numerical example may now be solved, and at the same time there will be explained other interesting phenomena caused by the closing of the gates and the pressure waves produced thereby. The analytical work may then be more readily understood.

Let $L = 820$ ft.;
$V_0 = 11.75$ ft. per sec.;
$H_0 = 165$ ft.;
$T = 2.1$ sec. (For convenience, T has been chosen an even multiple of $\frac{2L}{a}$);
$a = 4\,680$ ft. per sec. (This value of a is chosen simply because this example was worked out by the writer for a pipe in a tunnel and concreted in. The expansion of the pipe walls, therefore, was neglected. For any condition, a may be obtained by Equation (2)).

Since $V_0 = B_0 \sqrt{H_0}$, then $B_0 = 0.91476$.

Assume that the gates are closed in 24 successive instantaneous movements. The time elapsing between each movement would then be 0.0875 sec. After the first of these movements had taken place the gates would have been closed one-twenty-fourth of their opening, and the number representing the gate opening would have been reduced by one-twenty-fourth of its value, that is, $\frac{0.91476}{24} = 0.038115$. At each of the successive movements the value of B is reduced by the same amount, as the gate motion is assumed to be uniform. It will not be necessary, however, to use more than the first three significant figures, and the work may be done on the slide-rule. In this example the recovery of the friction head in the penstock will be neglected.

From the foregoing may now be written the first three columns of Table 1, and the first line of Columns 4 and 5. The table may then be completed as follows: Assume a trial reduction in velocity, caused by the initial instantaneous movement of the gate, and set the figure down in Column 5 under the value of V_0 and subtract it from V_0, placing

TABLE 1.—BY ARITHMETIC INTEGRATION.

Data: $L = 820$ ft. $V_0 = 11.75$ ft. per sec. $H_0 = 165$ ft. $T = 2.1$ sec.
$a = 4\,680$ ft. per sec. Friction neglected.

(1) Interval.	(2) Time, t.	(3) Gate, B.	(4) Head, H.	(5) Velocity, V.	(6) $145\,\Delta V$, Δh.	(7) $\Sigma(\Delta h)$, h_t.
0	0.0000	0 91476	165.00	11.75	......	......
		0.03811		0.085	12.32	......
¼	0.0875	0.87665	177.32	11.665		12.32
		0.03811		0.095	13.78	
½	0.1750	0.83854	191.10	11.570		26.10
		0.03811		0.10	14.50	
¾	0.2625	0.80043	205.60	11.47		40.60
		0.03811		0.112	16.20	
1	0.3500	0.76232	221.80	11.358		56.80
		0.03811		0.258	37.40	
1¼	0.4375	0.72421	234.56	11.10		69.56
		0.03811		0.29	42.10	
1½	0.5250	0.68610	249.10	10.81		84.10
		0.03811		0.30	43.50	
1¾	0.6125	0.64799	263.60	10.51		98.60
		0.03811		0.33	47.80	
2	0.7000	0.60988	279.00	10.18		114.00
		0.03811		0.43	62.30	
2¼	0.7875	0.57177	291.14	9.75		126.14
		0.03811		0.47	68.20	
2½	0.8750	0.53366	302.70	9.28		137.70
		0.03811		0.48	69.60	
2¾	0.9625	0.49555	314.30	8.80		149.30
		0.03811		0.53	76.80	
3	1.0500	0.45744	327.90	8.27		162.90
		0.03811		0.58	84.00	
3¼	1.1375	0.41933	337.46	7.69		172.46
		0.03811		0.61	88.40	
3½	1.2250	0.38111	346.10	7.08		181.10
		0.03811		0.62	90.00	
3¾	1.3125	0.34300	354.90	6.46		189.90
		0 03811		0.67	97.10	
4	1.4000	0.30489	361.60	5.79		196.60
		0.03811		0.68	98.60	
4¼	1.4875	0.26678	366.60	5.11		201.64
		0 03811		0.70	101.50	
4½	1.5750	0.22867	371.10	4.41		206.10
		0.03811		0.715	103.80	
4¾	1.6625	0.19056	376.10	3.695		211.10
		0.03811		0.730	106.00	
5	1.7500	0.15245	378.30	2.965		213.30
		0.03811		0.730	106.00	
5¼	1.8375	0.11434	380.66	2.235		215.66
		0.03811		0.745	108.00	
5½	1.9250	0.07623	382.70	1.490		217.70
		0.03811		0.745	108.00	
5¾	2.0125	0.03812	381.90	0.745		216.90
				0.745	108.00	
6	2.1000	0.0	381.70	0.0		216.70

the difference immediately underneath. This trial figure is ΔV, and is assumed to be destroyed instantaneously by the first movement of the gate. A pressure wave, Δh, of magnitude, $\frac{a\,\Delta v}{g} = \frac{4\,680}{32.2}(\Delta V) = 145\,\Delta V$, is therefore started up the pipe. The product of $145\,\Delta V$ is set down in Column 6 opposite ΔV. In Column 7 is recorded the algebraic sum of the values of Δh. Having obtained the figure in Column 7, it is added to the net head, $H_0 = 165$, and the sum is set down in the next line lower in Column 4. The result must now be checked, to see that $B\sqrt{H} = V$, where B is 0.8766 and H and V have the values recorded in their respective columns opposite B. If the relation is not satisfied, a new trial value of ΔV must be chosen, and the operations repeated until a check is obtained. After trial, the initial value of ΔV was found to be 0.085. Proceeding in this way, the rise of pressure at the end of 0.35 sec. is found to be 56.80 ft. This rise has taken place in four successive jumps.

At this point it becomes necessary to trace the course of the pressure wave started up the penstock by the initial movement of the gate. This wave has a velocity of 4 680 ft. per sec., and travels at that rate toward the forebay or origin of the penstock; after arriving at the origin it is reflected and returns to the gate at the same velocity. The distance from gate to forebay and return is 1 640 ft., so that the pressure wave takes 0.35 sec. to cover this distance. On its arrival at the gate it is reflected immediately as a wave of sub-normal pressure, and commences its journey again from gate to forebay and back. At the instant the wave becomes sub-normal, however, the gate is given one of its instantaneous closing movements, causing a further reduction in the velocity of the water flowing in the penstock and the consequent rise of pressure incident thereto. Thus, at this instant, two factors have to be taken into consideration: the rise of pressure caused by the fifth little instantaneous movement of the gate and the fall in pressure caused by the change from super-normal to sub-normal of the pressure wave produced by the first or initial movement of the gate which occurred 0.35 sec. before.

By trial-and-error the velocity that has been destroyed by the fifth movement of the gate is found to be 0.258 ft. per sec., and the result is checked as follows. Multiplying 0.258 by 145, the magnitude of the resulting pressure wave is 37.4 ft., and this added to 56.80 would make the total excess pressure existing equal to 94.20 ft., were it not for the fact that the initial wave has returned to the gate and become sub-normal. The amount of the initial wave, as shown by the second line of Table 1, is 12.32, and since it not only falls to zero but passes below zero to a sub-normal pressure of equal amount, there must be subtracted twice 12.32 from 94.20, making the net excess pressure existing at that instant 69.56 ft. Adding this to the initial net head, $H_0 = 165$, the value of V is checked as before by the formula, $V = B\sqrt{H}$.

Proceeding then as before, and remembering that the pressure rise caused by the sixth movement of the gate is reduced by the fall to sub-normal of the wave produced by the second movement of the gate, and the pressure rise of the seventh movement is reduced by the wave produced by the third movement, and so on, there may be calculated the successive increments of pressure.

For convenience, the time required by the pressure waves to travel from the gate to the origin of the pipe and return to the gate again will be called one interval. The interfering waves referred to in the foregoing paragraph are then always one interval apart.

After the eighth movement of the gate has taken place, it is found that the net excess pressure has reached 114.00 ft. At the instant the ninth movement takes place it must be noted that the pressure wave produced by the first movement of the gate, which has traveled, during the first interval, from the gate to the origin and back as a wave of super-normal pressure, and during the second interval over the same course as a wave of sub-normal pressure, has now returned to the gate and, again becoming super-normal, is again reflected and commences its journey from gate to origin and back. Thus, at the ninth movement of the gate, there must be taken into consideration the pressure wave caused by the instantaneous destruction of the velocity at that instant, the sub-normal wave due to the fifth movement, and the super-normal wave due to the first movement. After adding to the excess pressure existing at the end of the eighth movement, the rise of pressure caused by the ninth movement, there must be subtracted twice the pressure caused by the fifth movement, and there must be added twice the pressure caused by the first movement, or, in other words, there is subtracted twice the difference between the fifth and first waves.

By thus keeping in mind the position of the wave propagated by each movement of the gate, Table 1 may be completed, and the resulting maximum rise of pressure is found to be 217.70 ft.

Fig. 1 is a series of graphical diagrams showing the magnitude of the pressure waves caused by the successive instantaneous movements of the gate. A separate diagram is drawn for each movement, pressure being represented by the ordinates and time by the abscissas

The change from super-normal to sub-normal and *vice versa* is shown at the end of each interval of time. Fig. 2 is the excess pressure-time curve plotted from the figures in Column 7 of Table 1, or, what amounts to the same thing, from the algebraic sum of the pressure waves shown in Fig. 1.

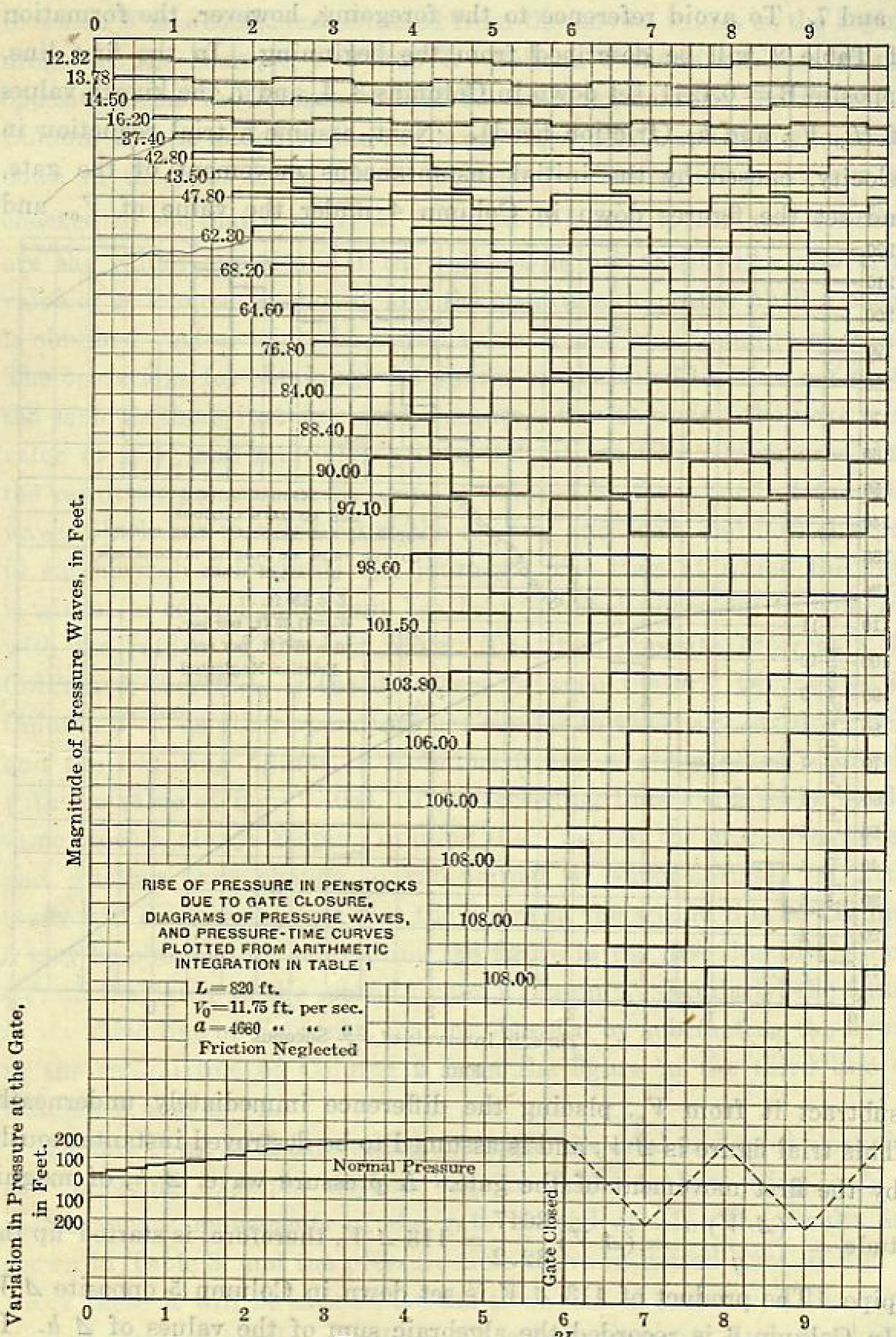
Table 2 shows the method of determining the results when the recovery of friction in the penstock is included, an example being selected where the friction head is an appreciable quantity. The

TABLE 2.—By Arithmetic Integration.

Data: $L = 6\,337$ ft. $V_0 = 15.055$ ft. per sec. $H_0 = 1\,260$ ft. $T = 69.5$ sec. $a = 3\,647$ ft. per sec. Friction head $h_f = 81$ ft.

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Interval.	Gate,* B.	Head, H.	Velocity, V.	$113\ \Delta V$, Δh.	$\Sigma\ \Delta h$.	$3\,574\ V^2$. h_f.	Δh_f.	$\Sigma(\Delta h + \Delta h_f)$, h_t
0	0.4241	1260.0	15.055		....	81.0	....	....
			0.155	17.5				
1	0.4168	1279.2	14.90		17.5	79.3	1.7	19.2
			0.29	32.8				
2	0.4084	1279.8	14.61		15.3	76.5	4.5	19.8
			0.28	31.7				
3	0.3995	1284.0	14.33		16.4	73.4	7.6	24.0
			0.33	37.3				
4	0.3895	1291.8	14.00		20.9	70.1	10.9	31.8
			0.37	41.8				
5	0.3784	1295.5	13.63		20.9	66.4	14.6	35.5
			0.42	47.5				
6	0.3657	1305.3	13.21		26.6	62.3	18.7	45.3
			0.49	55.4				
7	0.3513	1311.8	12.72		28.6	58.0	23.0	51.8
			0.53	59.8				
8	0.3354	1319.0	12.19		31.0	53.0	28.0	59.0
			0.60	67.8				
9	0.3180	1329.8	11.59		36.8	48.0	33.0	69.8
			0.65	73.5				
10	0.2991	1334.9	10.94		36.7	42.8	38.2	74.9
			0.72	81.3				
11	0.2781	1348.2	10.22		44.6	37.4	43.6	88.2
			0.84	94.8				
12	0.2544	1359.7	9.38		50.2	31.5	49.5	99.7
			0.92	104.0				
13	0.2285	1369.2	8.46		53.8	25.6	55.4	109.2
			0.97	109.5				
14	0.2009	1376.6	7.49		55.7	20.1	60.9	116.6
			1.07	121.0				
15	0.1719	1391.6	6.42		65.3	14.7	66.3	131.6
			1.15	130.0				
16	0.1411	1395.8	5.27		64.7	9.9	71.1	135.8
			1.21	136.8				
17	0.1083	1407.2	4.06		72.1	5.9	75.1	147.2
			1.33	150.1				
18	0.0725	1416.3	2.73		78.0	2.7	78.3	156.3
			1.365	154.2				
19	0.0363	1416.5	1.365		76.2	0.7	80.3	156.5
			1.365	154.2				
20	0.	1419.0	0.		78.0	0.	81.0	159.0

* Non-uniform gate motion.


0 1 2 3 4 5 6 7 8 9
12.32
13.78
14.50
16.20
37.40
42.80
43.50
47.80
62.30
68.20
64.60
76.80
84.00
88.40
90.00
97.10
98.60
101.50
103.80
106.00
106.00
108.00
108.00
108.00
Magnitude of Pressure Waves, in Feet.
RISE OF PRESSURE IN PENSTOCKS DUE TO GATE CLOSURE, DIAGRAMS OF PRESSURE WAVES, AND PRESSURE-TIME CURVES PLOTTED FROM ARITHMETIC INTEGRATION IN TABLE 1
L = 820 ft.
V0 = 11.75 ft. per sec.
a = 4660 " " "
Friction Neglected
Variation in Pressure at the Gate, in Feet.
200
100
0
100
200
Normal Pressure
Gate Closed
0 1 2 3 4 5 6 7 8 9
Time, in Intervals of 2L/a Seconds.
FIG. 1.

operations are almost identical with those just described, up to Columns 6 and 7. To avoid reference to the foregoing, however, the formation of Table 2 will be described from the beginning. In the first line, opposite B = 0.4241, set down in Columns 3, 4, and 7, the known values of H_0, V_0, and h_f (friction head). Next, assume a trial reduction in velocity, caused by the initial instantaneous movement of the gate, and set the figures down in Column 4 under the value of V_0, and

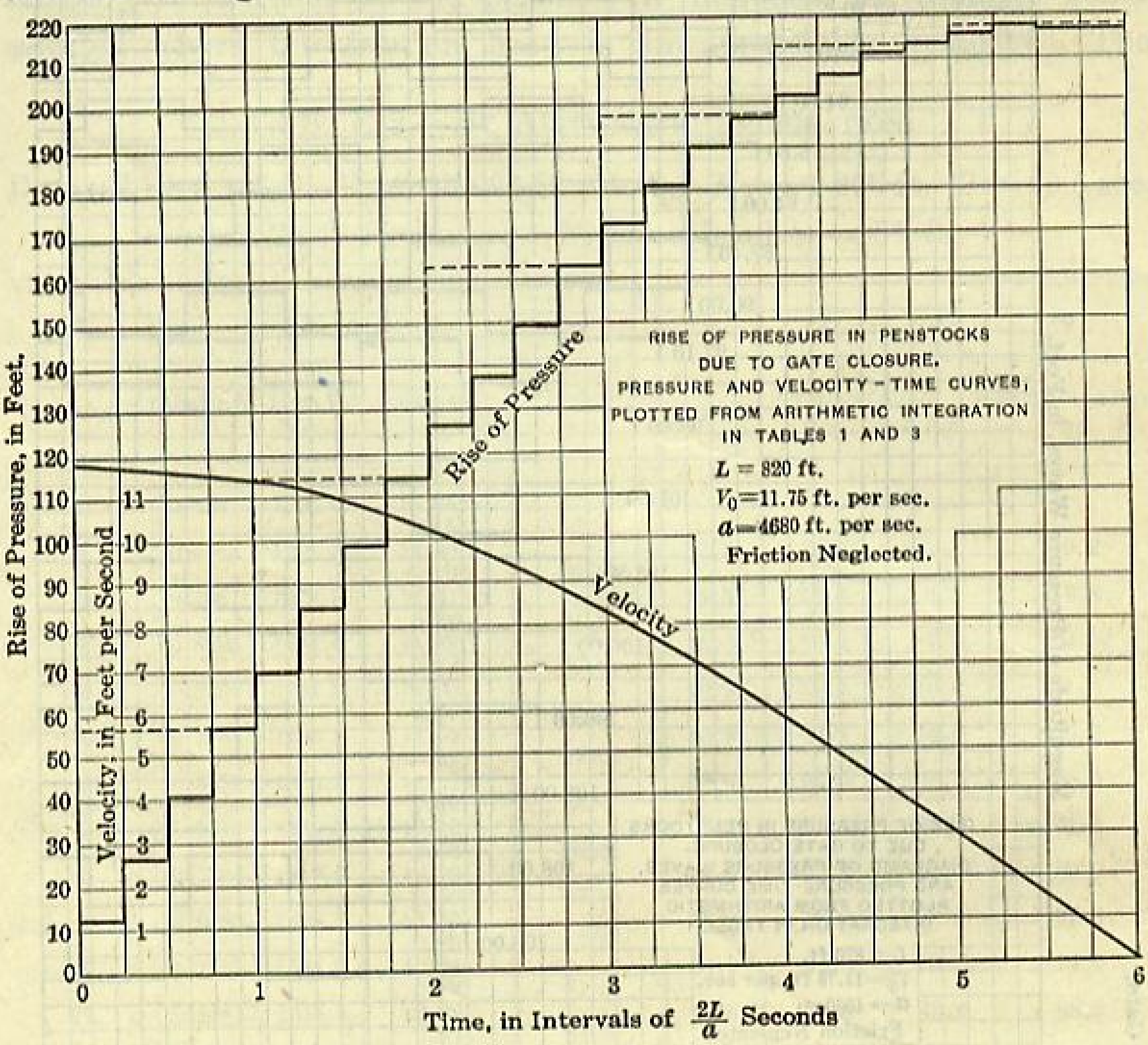


FIG. 2.

subtract it from V_0, placing the difference immediately underneath. This trial figure is ΔV, and is assumed to be destroyed instantaneously by the first movement of the gate. A pressure wave, Δh, of magnitude $= \frac{(\Delta V)\,a}{g} = (\Delta V)\frac{3647}{32.2} = 113\ \Delta V$, therefore, is started up the pipe. The product of 113 ΔV is set down in Column 5 opposite ΔV. In Column 6 is recorded the algebraic sum of the values of Δh. In Column 7 the total friction head, due to the velocity shown in Column 4, is set down, and, in Column 8, the friction head recovered at each operation is shown. For convenience, h_f may be made equal to $F\ V^2$, in which F is a coefficient obtained from the known values at the beginning. In the example, $F = 0.3574$. Column 9 shows the sum of the opposite items in Columns 6 and 8. Having obtained the figure in Column 9, it is added to the net head, $H_0 = 1\,260$, and the sum is set down in the next lower line in Column 3. The result must now be checked to see that $B \sqrt{H} = V$, where B is now 0.4168, and H and V are the values opposite. If the relation is not satisfied, a new trial value of $\Delta\ V$ must be chosen, and the operations repeated until a check is obtained. After trial, the initial value of $\Delta\ V$ was found to be 0.155. The operations for obtaining the figures on the third line are not quite the same as those just described, because, after assuming the next trial value of $\Delta\ V$, and multiplying it by 113, it must be remembered that the resulting pressure at the gate is reduced by the return of the first wave, which has traveled up to the forebay and back, and now changes to sub-normal and repeats the journey. The time at which the gate is given its second movement has been selected purposely to coincide with the return of the first wave. The item opposite Interval 2, in Column 6, therefore, is the difference between the first two figures in Column 5. The other operations are similar to those already described, and the resulting figures in Columns 3 and 4 are checked similarly with the value of $B = 0.4084$. The succeeding lines are filled in by the same process, always keeping in mind the return of the preceding waves, and whether they change to sub-normal or super-normal. A little study will disclose the fact that the figure in the second line of Column 6 may be obtained by subtracting the figure in the first line of Column 6 from the figure in the second line of Column 5. Similarly, the figure in the third line of Column 6 may be obtained by subtracting the figure in the second line of Column 6 from the figure in the third line of Column 5, and so on.

At this point it is interesting to repeat the trial-and-error work for the foregoing example worked out in Table 1, but using only 6 instantaneous movements of the gate instead of 24. The results are shown in Table 3, and the pressure-time curve in dotted lines in Fig. 2. From these it will be noted that the total rise of pressure is the same as that obtained by the calculations for 24 movements, and, moreover, the resulting pressure at the end of each interval is the same in both cases. Although six movements of the gate, or one movement to each

TABLE 3.—BY ARITHMETIC INTEGRATION.

For Data, see Table 1.

(1)	(2)	(3)	(4)	(5)	(6)	(7)
Interval.	Time, T.	Gate, B.	Head, H.	Velocity, V.	$145 \Delta V$, Δh.	$\Sigma (\Delta h)$, h_t.
0	0.0	0.91476	165.00	11.75		.....
				0.392	56.8	
1	0.35	0.762	221.8	11.358		56.8
				1.178	171.0	
2	0.70	0.608	279.2	10.18		114.2
				1.91	277.0	
3	1.05	0.457	327.8	8.27		162.8
				2.48	360.0	
4	1.40	0.304	362.20	5.79		197.2
				2.825	410.0	
5	1.75	0.1525	377.8	2.965		212.8
				2.965	430.0	
6	2.10	0.0	382.2	0.00		217.2

interval of time, are sufficient to determine the pressure rise at the end of each interval, it requires the larger number of movements to obtain intermediate points on the pressure-time curve. If a still greater number of movements is taken, the increments in pressure rise become smaller, and, in the limit, the stepped diagrams similar to Fig. 2 would become a series of smooth curves from the beginning to the end of each interval. The diagram, however, would not necessarily form a smooth curve from the beginning to the end of the closing time, because cusps or changes of curvature at the end of each interval result from the action of the pressure waves in changing at that instant from super-normal to sub-normal or *vice-versa.* When the duration of closure is short, the change of curvature in the diagram at the end of each interval is frequently very apparent; but, when the duration of closure is long, the changes of curvature in many cases cannot be detected by the eye. These changes of curvature at the end of the intervals make it difficult to formulate the integration which can be performed so easily by the trial-and-error work already explained.

It is possible, however, to obtain a series of equations, one for each interval of the closing time, that constitute a direct mathematical solution of the problem, without recourse to trial-and-error methods. The foregoing example has been explained in order that the analytical work may be more easily understood, and in order that the method of tracing the course of the pressure waves and keeping track of their periodic changes may be kept clearly in mind. More particularly, it should be borne in mind that though a formula may be written for the pressure rise in the first interval, using only the known quantities existing before the shut-down, the formulas for the pressure rise in any succeeding interval will involve, as one of the known quantities, the value of the pressure rise at the end of the preceding interval. It thus becomes necessary, when calculating the pressure rise at any time during, or at the end of, the closing of the gate, to obtain first the amount of the pressure rise at the end of each preceding interval. Moreover, if it is desired to determine the pressure rise at any time between the beginning and end of any interval—that is, at any fractional part of an interval—not only is it necessary to determine the pressure rise at the end of each preceding interval, but also the pressure rise at the same fractional part of each preceding interval. This also applies in determining the excess pressure-time curve when the duration of closure is not exactly a whole number of intervals.

NOMENCLATURE.

L = length of penstock, in feet;

a = velocity of pressure wave, in feet per second;

g = acceleration due to gravity, in feet per second per second;

$\frac{2L}{a}$ = one interval of time;

T = governor time, in seconds, *i. e.*, total duration of gate closure;

$T_1, T_2, T_3, \ldots T_n$ = time at the end of the 1st, 2d, 3d, ... nth interval;

$t_1, t_2, t_3, \ldots t_n$ = any time during the 1st, 2d, 3d, ... nth interval;

n = any number of intervals; the final interval in the time, T, need not be complete;

H_0 = normal net head, in feet;

$h_1, h_2, h_3, \ldots h_n$ = excess pressure above normal, in feet, existing at the end of the 1st, 2d, 3d, ... nth interval, *i. e.*, at the times, $T_1, T_2, T_3, \ldots T_n$;

$h_{t_1}, h_{t_2}, h_{t_3}, \ldots h_{t_n}$ = excess pressure above normal, in feet, at any time during the 1st, 2d, 3d, ...nth interval, *i. e.*, at the times, $t_1, t_2, t_3, \ldots t_n$;

V_0 = initial velocity, in feet per second, of the water in the penstock before shut-down.

$V_1, V_2, V_3, \ldots V_n$ = velocity, in feet per second, of the water in the penstock at the end of the 1st, 2d, 3d, ... nth interval;

$V_{t_1}, V_{t_2}, V_{t_3}, \ldots V_{t_n}$ = velocity, in feet per second, of the water in the penstock at any time, $t_1, t_2, t_3, \ldots t_n$, during the 1st, 2d, 3d, ... nth interval;

$B_0 = \frac{V_0}{\sqrt{H_0}}$ = a number representing the gate opening;

$$S_{t_1} = \left(\frac{a}{g}\right)^2\left(1 - \frac{t_1}{T}\right)^2 B_0^2 \qquad R_0 = \frac{a}{g} V_0$$

$$S_{t_2} = \left(\frac{a}{g}\right)^2\left(1 - \frac{t_2}{T}\right)^2 B_0^2 \qquad R_{t_1} = h_1 + \frac{a}{g} V_1 - C_{t_1}$$

$$S_{t_n} = \left(\frac{a}{g}\right)^2\left(1 - \frac{t_n}{T}\right)^2 B_0^2 \qquad R_{t_2} = h_2 + \frac{a}{g} V_2 - C_{t_2}$$

$$C_{t_1} = 2\frac{a}{g}(V_0 - V_{t_1}) \qquad R_{t_{n-1}} = h_{n-1} + \frac{a}{g} V_{n-1} - C_{t_{n-1}}$$

$$C_{t_2} = 2\frac{a}{g}(V_1 - V_{t_2}) - C_{t_1} \qquad Z_1 = \frac{\left(\frac{a}{g}\right)^2 S_{t_1}}{\left(\frac{a}{g}\right)^2 + S_{t_1} F}$$

$$C_{t_n} = 2\frac{a}{g}(V_{n-1} - V_{t_n}) - C_{t_{n-1}}$$

F = a friction factor such that $F V^2$ = total loss, in feet of head, for velocity, V;

$$Z_2 = \frac{\left(\frac{a}{g}\right)^2 S_{t_2}}{\left(\frac{a}{g}\right)^2 + S_{t_2} F}$$

$$Z_n = \frac{\left(\frac{a}{g}\right)^2 S_{t_n}}{\left(\frac{a}{g}\right)^2 + S_{t_n} F}$$

Time is always to be measured from the beginning of the gate movement. Thus, the time, t_2, is the time from the beginning of the stroke to some time between the first and third intervals. The successive values of t_1, t_2, etc., must always be one interval apart.

First: Derivation of Formulas with Friction Neglected.—Joukovsky's formula for water-hammer, as already stated in Equation (1), is:

$$h = \frac{a\,V}{g}$$

During slow closing of the gate, $dh = \frac{a}{g} dV$.

At the beginning of the closing time, as already stated in Equation (3),

$$V_0 = B_0 \sqrt{H_0}$$

and, in general, $V = B\sqrt{H_0 + h}$.

For the first interval:

$$\int_0^{h_{t_1}} dh = \frac{a}{g}\int_{V_{t_1}}^{V_0} dV$$

where h_{t_1} = pressure rise at any time, t_1, during the first interval, and V_{t_1} = velocity at the time, t_1, during the first interval.

Integrating, then:

$$h_{t_1} = \frac{a}{g}(V_0 - V_{t_1}) \dots\dots\dots\dots (5)$$

Since V_{t_1} must be proportional to the gate opening multiplied by the square root of the head, then, for uniform gate motion:

$$V_{t_1} = \left(1 - \frac{t_1}{T}\right) B_0 \sqrt{H_0 + h_{t_1}} \dots\dots\dots\dots (6)$$

Substituting this value of V_{t_1} in the equation for h_{t_1}, the latter becomes:

$$h_{t_1} = \frac{a}{g} V_0 - \frac{a}{g}\left(1 - \frac{t_1}{T}\right) B_0 \sqrt{H_0 + h_{t_1}}$$

For simplicity, substitute the symbol, R_0, in place of $\frac{a}{g} V_0$, and the symbol, $\sqrt{S_{t_1}}$, for $\frac{a}{g}\left(1 - \frac{t_1}{T}\right) B_0$.

Thus: $h_{t_1} = R_0 - \sqrt{S_{t_1}(H_0 + h_{t_1})}$

Squaring: $(h_{t_1} - R_0)^2 = S_{t_1}(H_0 + h_{t_1})$

Expanding: $h_{t_1}^2 - 2h_{t_1} R_0 + R_0^2 = S_{t1} H_0 + S_{t_1} h_t$

Collecting: $h_{t_1}^2 - h_{t_1}(S_{t_1} + 2R_0) + R_0^2 - S_{t_1} H_0 = 0$

Solving: $h_{t_1} = \frac{1}{2}\{(S_{t_1} + 2R_0)$

$$\pm \sqrt{(S_{t_1} + 2R_0)^2 - 4R_0^2 + 4S_{t_1} H_0}\}$$

Simplifying: $h_{t_1} = \frac{1}{2}\{(S_{t_1} + 2R_0) \pm \sqrt{S_{t_1}(S_{t_1} + 4R_0 + 4H_0)}\} \dots (7)$

For the second interval, an examination of Table 1 will show that

$$\int_0^{h_{t_2}} dh = \frac{a}{g}\int_{V_{t_2}}^{V_1} dV + h_1 - 2\frac{a}{g}\int_{V_{t_1}}^{V_0} dV$$

Where h_{t_2} = pressure rise at any time, t_2, during the second interval;
V_{t_2} = velocity at the time, t_2;
V_1 = velocity at the end of the first interval;
h_1 = pressure rise at the end of the first interval;
V_{t_1} = velocity at the time, t_1, which must be exactly one interval before the time, t_2.

Integrating: then $h_{t_2} = \frac{a}{g}(V_1 - V_{t_2}) + h_1 - 2\frac{a}{g}(V_0 - V_{t_1})$....(8)

Substituting C_{t_1} for $2\frac{a}{g}(V_0 - V_{t_1})$

$$h_{t_2} = \frac{a}{g}(V_1 - V_{t2}) + h_1 - C_{t_1}$$

Again, since V_{t_2} must be proportional to the gate opening multiplied by the square root of the head, then, for uniform gate motion:

$$V_{t_2} = \left(1 - \frac{t_2}{T}\right) B_0 \sqrt{H_0 + h_{t_2}} \text{..............(9)}$$

Substituting this value of V_{t_2} in the equation for h_{t_2} the latter becomes

$$h_{t_2} = \frac{a}{g} V_1 - \frac{a}{g}\left(1 - \frac{t_2}{T}\right) B_0 \sqrt{H_0 + h_{t_2}} + h_1 - C_{t_1}.$$

Substituting the symbol, R_{t_1}, in place of $\frac{a}{g} V_1 + h_1 - C_{t_1}$, and the symbol, $\sqrt{S_{t_2}}$, for $\frac{a}{g}\left(1 - \frac{t_2}{T}\right) B_0$,

$$h_{t_2} = R_{t_1} - \sqrt{S_{t_2}(H_0 + h_{t_2})}$$

Squaring: $(h_{t_2} - R_{t_1})^2 = S_{t_2}(H_0 + h_{t_2})$

Expanding: $h_{t_2}{}^2 - 2\, h_{t_2} R_{t_1} + R_{t_1}{}^2 = S_{t_2} H_0 + S_{t_2} h_{t_2}$

Collecting: $h_{t_2}{}^2 - h_{t_2}(S_{t_2} + 2\, R_{t_1}) + R_{t_1}{}^2 - S_{t_2} H_0 = 0$

Solving: $h_{t_2} = \frac{1}{2}\left\{(S_{t_2} + 2\, R_{t_1}) \pm \sqrt{(S_{t_2} + 2\, R_{t_1})^2 - 4\, R_{t_1}{}^2 + 4\, S_{t_2} H_0}\right\}$

Simplifying: $h_{t_2} = \frac{1}{2}\left\{(S_{t_2} + 2\, R_{t_1}) \pm \sqrt{S_{t_2}(S_{t_2} + 4\, R_{t_1} + 4\, H_0)}\right\}$..(10)

In a similar manner, the value of h, at any time in any interval, may be found.

Second: Derivation of Formulas with Friction Included.—In the foregoing analysis no account has been taken of the effect of frictional losses of head in the penstock. As the velocity in the penstock is gradually destroyed, the friction head is gradually recovered and is added to the net head, producing discharge through the turbine gates. In any penstock the total frictional losses may be assumed to be proportional to the square of the velocity of flow, and may be represented by a coefficient, F, such that FV^2 = total loss in feet of head for a velocity, V. The velocity head recovered at any time in the first interval, therefore, is $F\,V_0^2 - F\,V_{t_1}^2$. This head must then be added to the sum of $H_0 + h_{t_1}$, in determining the relation between velocity, gate opening, and head, as given by Equation (6).

Thus: $$V_{t_1} = \left(1 - \frac{t_1}{T}\right) B_0 \sqrt{H_0 + h_{t_1} + F\,V_0^2 - F\,V_{t_1}^2}$$

Squaring: $$V_t^2 = \left(1 - \frac{t_1}{T}\right)^2 B_0^2 \left(H_0 + h_{t_1} + F\,V_0^2 - F\,V_{t_1}^2\right)$$

Collecting: $$V_{t_1}^2 \left\{1 + \left(1 - \frac{t_1}{T}\right)^2 B_0^2\,F\right\} = \left(1 - \frac{t_1}{T}\right)^2 B_0^2 \left(H_0 + h_{t_1} + F\,V_0^2\right)$$

Solving, and multiplying numerator and denominator by $\frac{a}{g}$:

$$V_{t_1} = \sqrt{\frac{\left(\frac{a}{g}\right)^2 \left(1 - \frac{t_1}{T}\right)^2 B_0^2 \left(H_0 + h_{t_1} + F\,V_0^2\right)}{\left(\frac{a}{g}\right)^2 + \left(\frac{a}{g}\right)^2 \left(1 - \frac{t_1}{T}\right)^2 B_0^2\,F}}$$

Substituting S_{t_1} for $\left(\frac{a}{g}\right)^2 \left(1 - \frac{t_1}{T}\right)^2 B_0^2$,

$$V_{t_1} = \sqrt{\frac{S_{t_1} \left(H_0 + h_{t_1} + F\,V_0^2\right)}{\left(\frac{a}{g}\right)^2 + S_{t_1} F}} \quad \ldots\ldots\ldots\ldots\ldots (11)$$

Inserting this value of V_{t_1} in Equation (5) for h_{t_1},

Then: $$h_{t_1} = \frac{a}{g} V_0 - \frac{a}{g} \sqrt{\frac{S_{t_1} \left(H_0 + h_{t_1} + F\,V_0^2\right)}{\left(\frac{a}{g}\right)^2 + S_{t_1} F}}$$

Solving for h_{t_1} in the same manner as that used in obtaining Equation (7), and substituting R_0 in place of $\frac{a}{g} V_0$, and Z_1 in place of

$$\frac{\left(\frac{a}{g}\right)^2 S_{t_1}}{\left(\frac{a}{g}\right)^2 + S_{t_1} F},$$

Then: $h_{t_1} = \frac{1}{2}\left\{(2\,R_0 + Z_1) \pm \sqrt{Z_1^2 + 4\,Z_1\,(R_0 + H_0 + F\,V_0^2)}\right\}$ ..(12)

In a similar manner, the equation for h_{t_2} is obtained by adding $F\,V_0^2 - F\,V_{t_2}^2$ under the root sign in Equation (9) and inserting the resulting value of V_{t_2} in Equation (8). Without performing the operations, which are similar to the above, the result may be written down at once.

$$h_{t_2} = \frac{1}{2}\left\{(2\,R_{t_1} + Z_2) \pm \sqrt{Z_2^2 + 4\,Z_2\,(R_{t_1} + H_0 + F\,V_0^2)}\right\} \quad ..(13)$$

Where: $R_{t_1} = \left(\frac{a}{g}\,V_1 + h_1 - C_{t_1}\right)$

$$C_{t_1} = \frac{2\,a}{g}(V_0 - V_{t_1})$$

$$Z_2 = \frac{\left(\frac{a}{g}\right)^2 S_{t_2}}{\left(\frac{a}{g}\right)^2 + S_{t_2}\,F}$$

and $S_{t_2} = \left(\frac{a}{g}\right)^2\left(1 - \frac{t_2}{T}\right)^2 B_0^2$

Similarly, the value of h, at any time in any interval, may be found.

When the gate motion is not uniform, the value of S_t has to be made equal to $x^2\left(\frac{a}{g}\right)^2 B_0^2\left(1 - \frac{t}{T}\right)^2$, where x is a constant or variable coefficient, which may be determined by plotting the curve of gate opening on a time base. For uniform motion, the curve would be a straight line, the ordinate of which, at any time, t, is $\left(1 - \frac{t}{T}\right)B_0$. For any other than uniform motion, the ordinates of the straight line would be multiplied by the constant or variable, x, which, if the motion varied in a regular manner, might sometimes be expressed in terms of t, either graphically or analytically, from the known relation between the governor movements and the gate opening. If the motion were not regular, a graphical solution only could be obtained.

Tables 4 and 5 show the values of h_t obtained by using the formulas in the examples worked out by arithmetic integration in Tables 1, 2, and 3.

It should be noted here, by way of caution, that, in calculating the value of h, the two terms inside the larger brackets of the foregoing formulas are frequently so large and their difference so small as to make it necessary to perform the work by logarithms. The correct result cannot be obtained in such cases by using an ordinary 10-in. slide-rule.

TABLE 4.—By Formulas.

Data: $L = 820$ ft. $V_0 = 11.75$ ft. per sec. $H_0 = 165$ ft. $T = 2.1$ sec. $a = 4\,680$ ft. per sec. Friction neglected.

(1) Interval.	(2) Gate, B.	(3) Rise of pressure, h_t.	(4) Velocity, V.
0	0.9148	0.0	11.7500
¼	0.8766	12.12	11.6664
½	0.8385	25.53	11.5740
¾	0.8004	40.41	11.4713
1	0.7623	56.96	11.3571
1¼	0.7242	69.94	11.1004
1½	0.6861	83.72	10.8205
1¾	0.6480	98.31	10.5149
2	0.6099	113.63	10.1805
2¼	0.5718	125.78	9.7505
2½	0.5337	138.05	9.2909
2¾	0.4956	150.32	8.8005
3	0.4574	162.42	8.2766
3¼	0.4193	171.97	7.6970
3½	0.3811	181.10	7.0899
3¾	0.3430	189.43	6.4574
4	0.3049	196.77	5.7994
4¼	0.2668	202.49	5.1146
4½	0.2287	207.14	4.4118
4¾	0.1906	211.00	3.6959
5	0.1525	213.79	2.9680
5¼	0.1143	215.75	2.2303
5½	0.0762	216.71	1.4887
5¾	0.0381	216.95	0.7446
6	0.0	216.57	0.0

It is now interesting and instructive to compare the results given by the foregoing formulas with those given by other formulas which have been proposed to determine the rise of pressure caused by gradually stopping the flow of water in pipes. For this purpose, Fig. 3 has been prepared. There are three such fundamentally different formulas known to the writer, namely, those of Mr. L. Allievi,* M. M. Warren,†

*The writer was unable to obtain an English translation of Mr. Allievi's paper, but his formula is commonly given as follows:

$$h = \frac{N H}{2} + H \sqrt{\frac{N^2}{4} + N}$$

Where h = rise of pressure, in feet, above normal;
H = normal net head, in feet;

$$N = \left(\frac{L V}{g T H}\right)^2;$$

L = length of penstock, in feet;
V = velocity of water in penstock, in feet per second;
T = duration of gate closure, in seconds;
g = acceleration due to gravity, in feet per second per second.

† *Transactions*, Am. Soc. C. E., Vol. LXXIX, pp. 238, 242. Mr. Warren's formula is:

$$h = \frac{L V}{g\left(T - \frac{L}{a}\right)},$$ where a = velocity of the pressure wave, and the other symbols have the same significance as in this paper.

TABLE 5.—BY FORMULAS.

Data: $L = 6\,337$ ft. $V_0 = 15.055$ ft. per sec. $H_0 = 1\,260$ ft. $T = 69.5$ sec. $a = 3\,647$ ft. per sec. Friction head, $h_f = 81$ ft.

Non-uniform Gate Motion.

(1) Interval.	(2) Gate, B.	(3) Rise of pressure.	(4) Velocity.
0	0.4241	0.	15.0550
1	0.4168	18.68	14.9040
2	0.4084	20.42	14.6137
3	0.3995	25.04	14.3210
4	0.3895	30.74	13.9931
5	0.3784	36.52	13.6251
6	0.3657	44.12	13.2063
7	0.3513	52.07	12.7250
8	0.3354	59.96	12.1854
9	0.3180	68.27	11.5898
10	0.2991	76.53	10.9388
11	0.2781	87.85	10.2097
12	0.2544	99.30	9.3792
13	0.2285	110.13	8.4551
14	0.2009	118.88	7.4606
15	0.1719	128.15	6.4024
16	0.1411	136.68	5.2746
17	0.1083	146.75	4.0599
18	0.0725	156.94	2.7301
19	0.0363	156.13	1.3638
20	0.0	159.21	0.

Assoc. M. Am. Soc. C. E., and H. C. Vensano,* M. Am. Soc. C. E. The formula by Allievi gives the maximum excess pressure caused by stopping the flow of a column of water, assumed as an incompressible fluid, moving in an inextensible pipe, and neglecting the effect of friction. Mr. R. D. Johnson,† Mr. A. H. Gibson,‡ and W. F.

* *Transactions*, Am. Soc. C. E., Vol. LXXIX, pp. 289–299; and Vol. LXXXII, p. 185. Mr. Vensano's formula is:

$h = \frac{2\,l\,V}{g\,T}$, with the limitation that h can never be greater than $\frac{V\,a}{g}$.

† *Transactions*, Am. Soc. C. E., Vol. LXXIX, pp. 277–281. Mr. Johnson's formula is:

$$H_{\text{max.}} = \frac{2\,M\,Y}{N^2}\left(M + \sqrt{M^2 + N^2}\right)$$

where $H_{\text{max.}}$ = maximum rise of pressure, in feet, above normal; $M = L\,V$; and $N = 2\,g\,Y\,T$, in which Y is the normal net head in feet, and the other symbols have the same significance as in this paper.

‡ "Water Hammer in Hydraulic Pipe Lines", by A. H. Gibson. Mr. Gibson's formula is: (In order to prevent confusion with the foregoing, some changes have been made in the nomenclature):

$$p' = \frac{w}{g}\left[\left(\frac{L}{A}\cdot\frac{A_1}{T}\right)^2 + \frac{L}{A}\cdot\frac{A_1}{T}\sqrt{2\,g\,H + \left(\frac{L}{A}\cdot\frac{A_1}{T}\right)^2}\right]$$

where p' = rise in pressure, in pounds per square foot, behind the valve at the instant when closure is complete, and therefore when p is maximum;

A = cross-sectional area of penstock, in square feet;

A_1 = maximum effective area of valve opening, in square feet;

w = weight of a cubic foot of water;

and the other symbols have the same significance as in this paper.

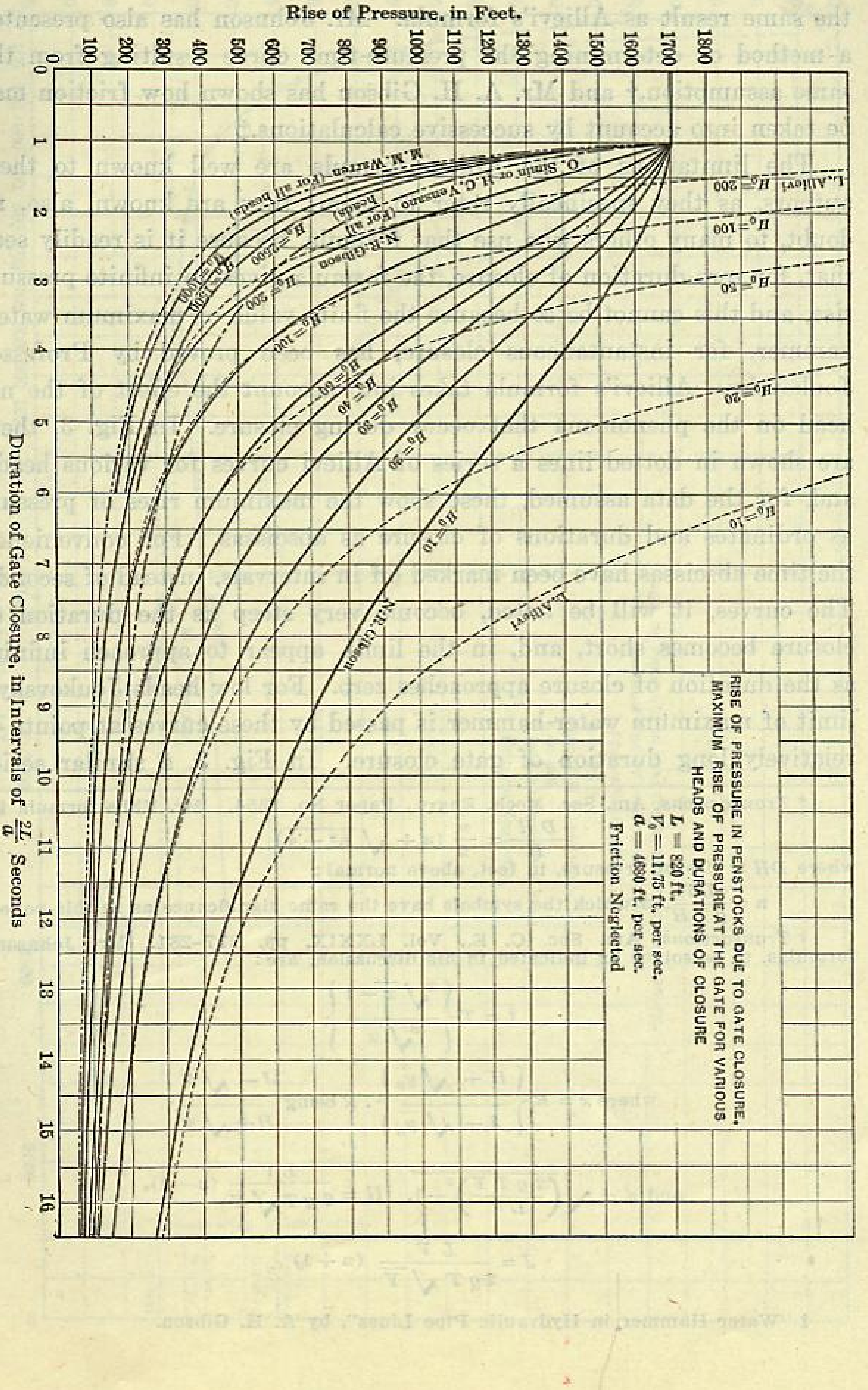

Rise of Pressure, in Feet.
0
100
200
300
400
500
600
700
800
900
1000
1100
1200
1300
1400
1500
1600
1700
1800
Duration of Gate Closure, in Intervals of $\frac{2L}{a}$ Seconds
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
M.M.Warren (For all heads)
O. Simin or H.C.Vensano (For all heads)
N.R.Gibson
L.Allievi
$H_o=200$
$H_o=100$
$H_o=50$
$H_o=20$
$H_o=10$
$H_o=2500$
$H_o=1000$
$H_o=500$
$H_o=40$
$H_o=30$
RISE OF PRESSURE IN PENSTOCKS DUE TO GATE CLOSURE,
MAXIMUM RISE OF PRESSURE AT THE GATE FOR VARIOUS
HEADS AND DURATIONS OF CLOSURE
$L = 820$ ft.
$V_o = 11.75$ ft. per sec.
$a = 4080$ ft. per sec.
Friction Neglected

FIG. 3.

Uhl,* M. Am. Soc. C. E., have each presented a similar formula, with different nomenclature and somewhat differently arranged, but all give the same result as Allievi's formula. Mr. Johnson has also presented a method of determining the pressure-time curve resulting from the same assumption,† and Mr. A. H. Gibson has shown how friction may be taken into account by successive calculations.‡

The limitations of the Allievi formula are well known to these authors, as they specifically refer to them; they are known, also, no doubt, to many others who use that formula, because it is readily seen that, for zero duration of closure, the formula gives an infinite pressure rise, and this cannot be so because the finite value of maximum water-hammer, for instantaneous closure, has been proved by Professor Joukovsky. Allievi's formula takes into account the effect of the net head on the phenomena that occur during closure. In Fig. 3, there are shown in dotted lines a series of Allievi curves for various heads, and, for the data assumed, these show the maximum rises of pressure as ordinates and durations of closure as abscissas. For convenience, the time abscissas have been marked off in intervals, instead of seconds. The curves, it will be noted, become very steep as the duration of closure becomes short, and, in the limit, appear to approach infinity as the duration of closure approaches zero. For low heads Joukovsky's limit of maximum water-hammer is passed by these curves at points of relatively long duration of gate closure. In Fig. 3, a similar series

* *Transactions*, Am. Soc. Mech. Engrs., Paper No. 1354. Mr. Uhl's formula is:

$$\frac{DH}{H} = \frac{n}{2}\left(n + \sqrt{n^2 + 4}\right)$$

where DH = rise of pressure, in feet, above normal;

$n = \frac{LV}{gTH}$, in which the symbols have the same significance as in this paper.

† *Transactions*, Am. Soc. C. E., Vol. LXXIX, pp. 277–281. Mr. Johnson's formulas, to be solved as indicated in his discussion, are:

$$t = T\left\{\frac{\sqrt[n]{x} - 1}{\sqrt[n]{x}}\right\}$$

$$\text{where } x = R\left\{\frac{H + \sqrt{y_0}}{J - \sqrt{y_0}}\right\}, \ R \text{ being } \frac{J - \sqrt{Y}}{H + \sqrt{Y}},$$

$$\text{and } n = \sqrt{\left(\frac{2gTY}{LV}\right)^2 + 1}, \quad H = \frac{LV}{2gT\sqrt{Y}}\,(n - 1),$$

$$J = \frac{LV}{2gT\sqrt{Y}}\,(n + 1)$$

‡ "Water Hammer in Hydraulic Pipe Lines", by A. H. Gibson.

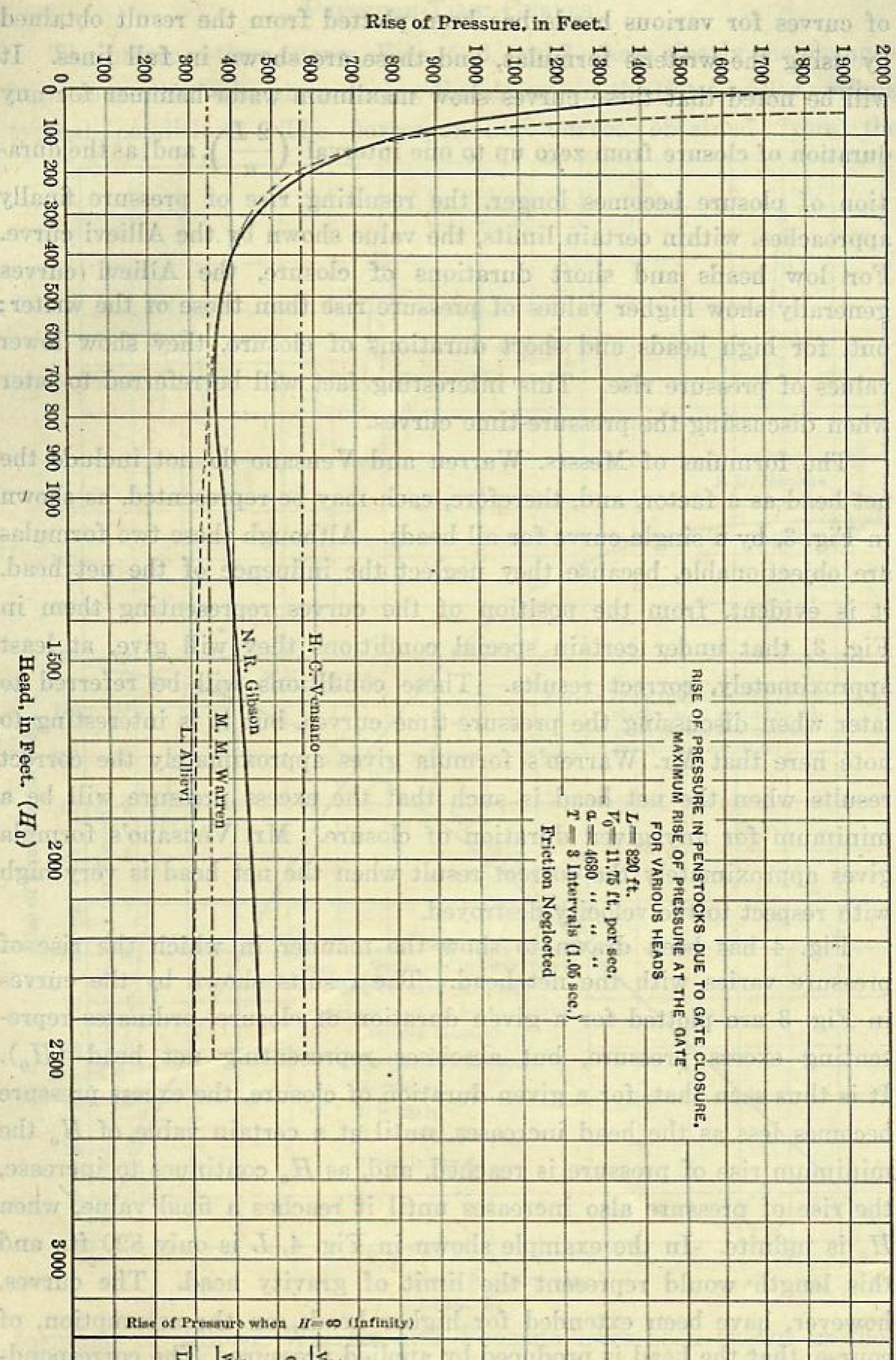

Rise of Pressure, in Feet.
RISE OF PRESSURE IN PENSTOCKS DUE TO GATE CLOSURE.
MAXIMUM RISE OF PRESSURE AT THE GATE
FOR VARIOUS HEADS
L = 820 ft.
V_0 = 11.75 ft. per sec.
a = 4680 " "
T = 3 Intervals (1.05 sec.)
Friction Neglected
H. C. Vensano
N. R. Gibson
M. M. Warren
L. Allievi
Head, in Feet. (H_0)
Rise of Pressure when H = ∞ (Infinity)

Fig. 4.

of curves for various heads has been plotted from the result obtained by using the writer's formulas, and these are shown in full lines. It will be noted that these curves show maximum water-hammer for any duration of closure from zero up to one interval $\left(\frac{2L}{a}\right)$, and, as the duration of closure becomes longer, the resulting rise of pressure finally approaches, within certain limits, the value shown by the Allievi curve. For low heads and short durations of closure, the Allievi curves generally show higher values of pressure rise than those of the writer; but, for high heads and short durations of closure, they show lower values of pressure rise. This interesting fact will be referred to later when discussing the pressure-time curves.

The formulas of Messrs. Warren and Vensano do not include the net head as a factor, and, therefore, each may be represented, as shown in Fig. 3, by a single curve for all heads. Although these two formulas are objectionable, because they neglect the influence of the net head, it is evident, from the position of the curves representing them in Fig. 3, that under certain special conditions they will give, at least approximately, correct results. These conditions will be referred to later when discussing the pressure-time curves, but it is interesting to note here that Mr. Warren's formula gives approximately the correct results when the net head is such that the excess pressure will be a minimum for any given duration of closure. Mr. Vensano's formula gives approximately the correct result when the net head is very high with respect to the velocity destroyed.

Fig. 4 has been drawn to show the manner in which the rise of pressure varies with the net head. The results shown by the curves in Fig. 3 are plotted for a given duration of closure, ordinates representing excess pressure, but abscissas representing net head (H_0). It is thus seen that, for a given duration of closure, the excess pressure becomes less as the head increases, until at a certain value of H_0 the minimum rise of pressure is reached, and, as H_0 continues to increase, the rise of pressure also increases until it reaches a final value, when H_0 is infinite. In the example shown in Fig. 4, L is only 820 ft., and this length would represent the limit of gravity head. The curves, however, have been extended for higher heads, on the assumption, of course, that the head is produced by applied pressure. The corresponding curves obtained from the results of the Allievi, Warren, and Vensano formulas are also clearly shown in Fig. 4.

Pressure-Time Curves.

The pressure-time curves, Figs. 5, 6, and 7, show clearly the changes in pressure that take place during the closure of the gates under various typical conditions. The corresponding curves obtained from the

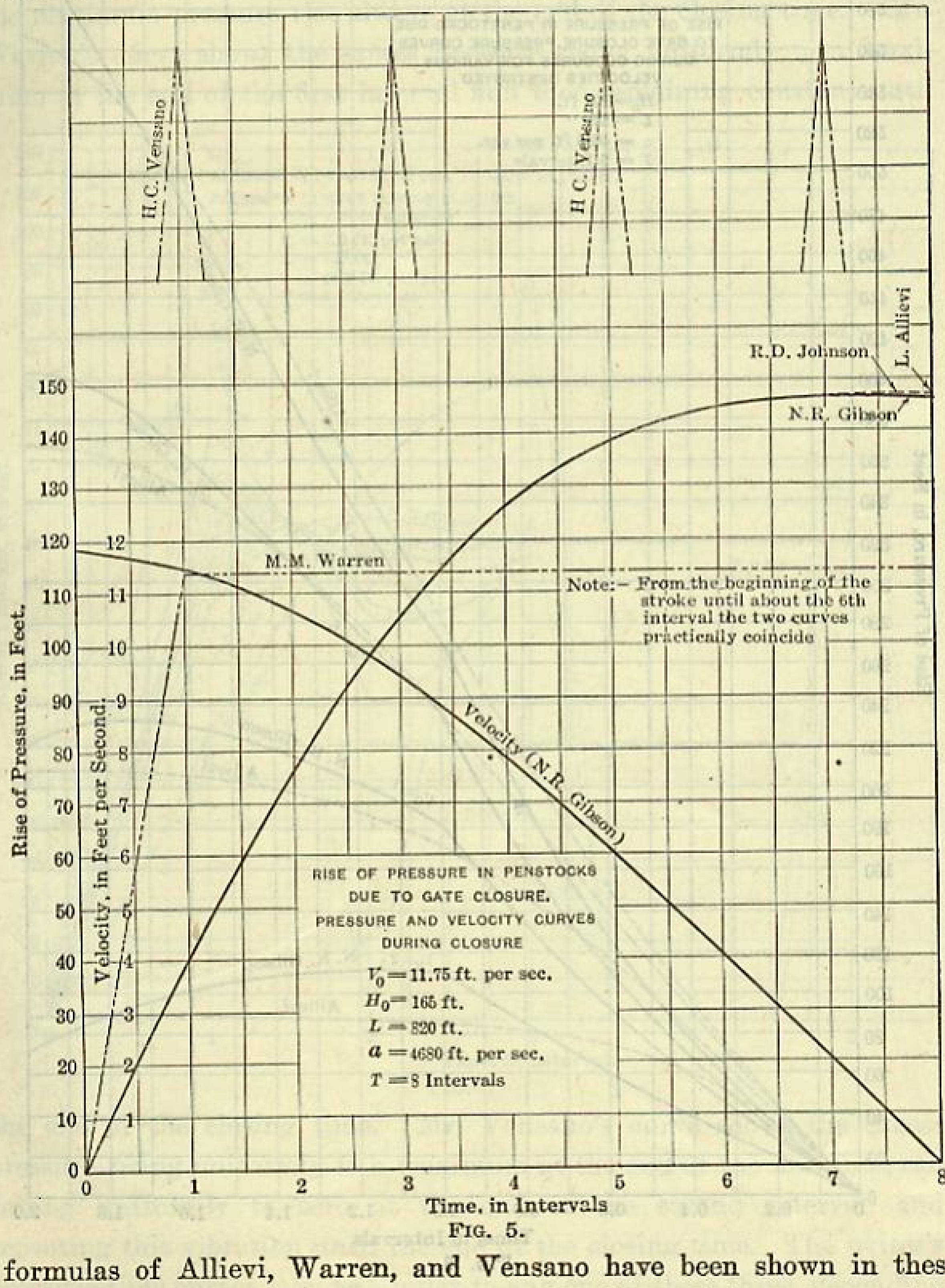


Fig. 5.

formulas of Allievi, Warren, and Vensano have been shown in these figures for the purpose of comparison with those of the writer, and in order that the various results may be studied and explained. In some

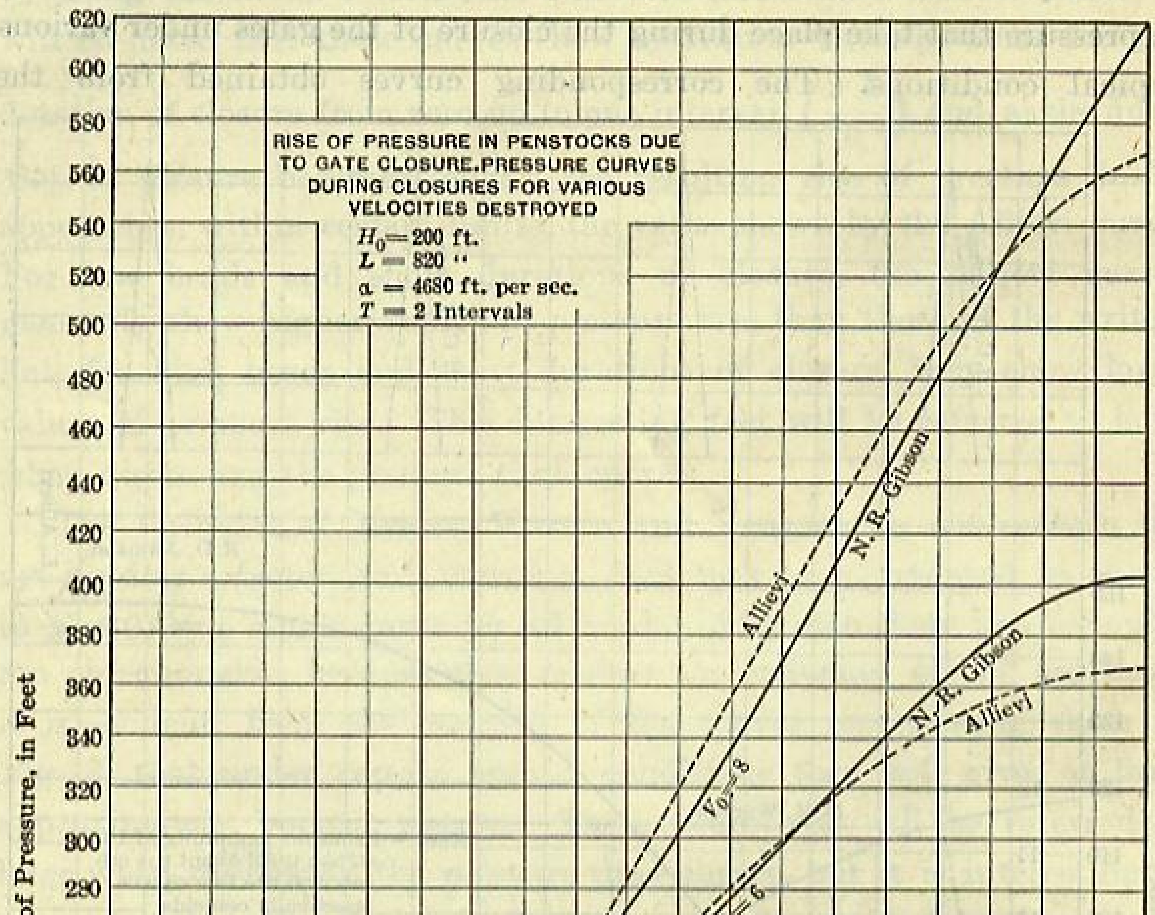

RISE OF PRESSURE IN PENSTOCKS DUE TO GATE CLOSURE. PRESSURE CURVES DURING CLOSURES FOR VARIOUS VELOCITIES DESTROYED
H_0 = 200 ft.
L = 820 "
a = 4680 ft. per sec.
T = 2 Intervals
N. R. Gibson
Allievi
V_0 = 8
V_0 = 6
V_0 = 4
V_0 = 2
Rise of Pressure, in Feet
Time, in Intervals

FIG. 6.

cases velocity-time curves have also been drawn, in order to show the variable rate of retardation caused by uniform gate closure.

It will be noted first that Mr. R. D. Johnson's pressure-time curve, based on Allievi's formula, or indeed Allievi's formula itself, gives the maximum pressure rise always at the end of the closing time. Mr. Warren's curve shows the excess pressure rising uniformly to a maximum at the end of the first interval and then remaining constant until

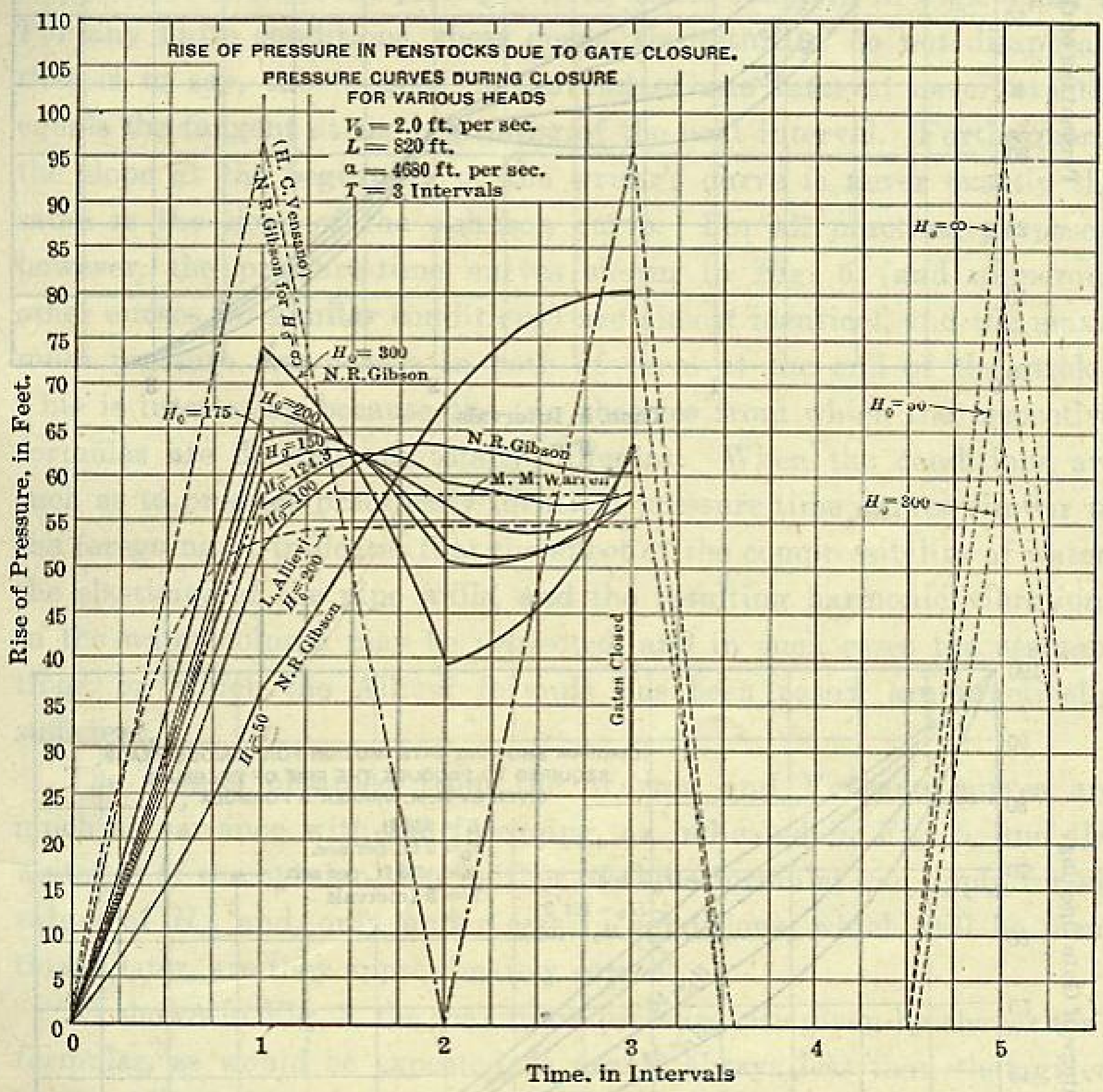


FIG. 7.

the end of the closing time. Mr. Vensano's curve shows the excess pressure rising uniformly to a maximum at the end of the first interval, falling uniformly to zero at the end of the second interval, and repeating this vibration until the end of the closing time. The writer's pressure-time curves may be similar to any one of these three, depending on the duration of closure and the net head acting on the orifice, or on both of these factors.

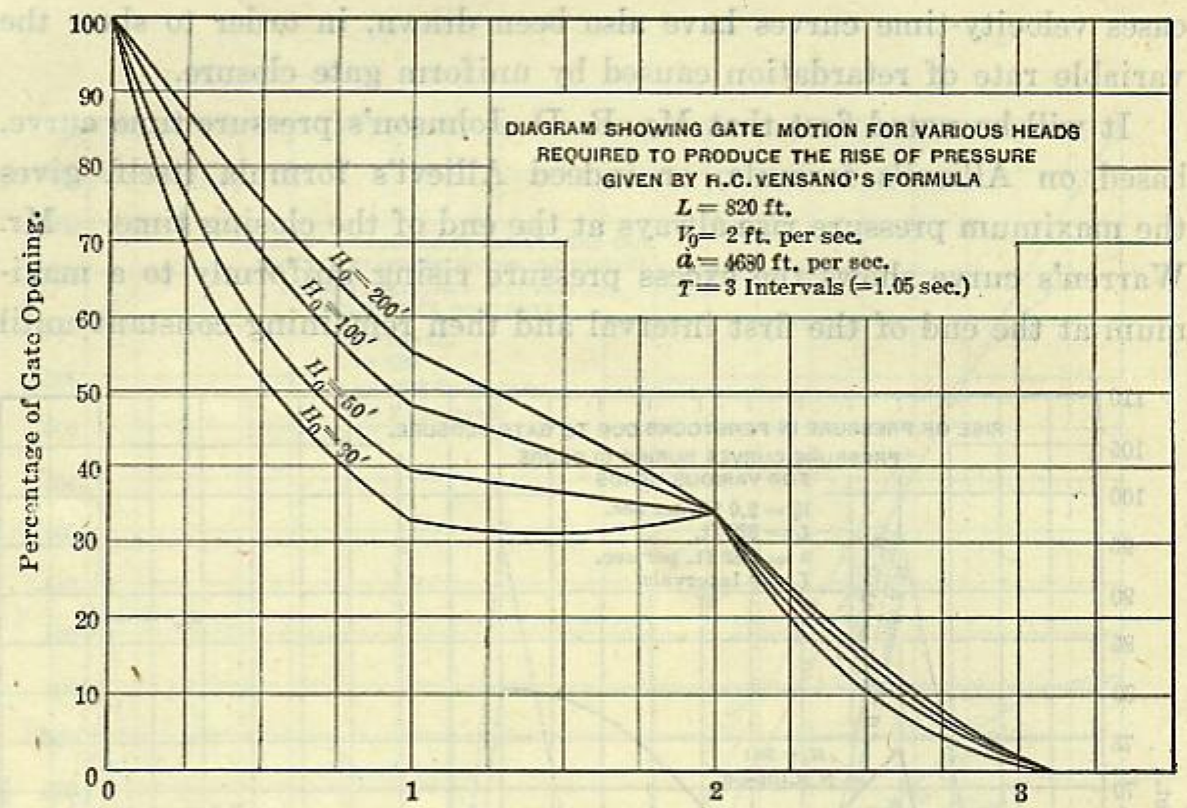


DIAGRAM SHOWING GATE MOTION FOR VARIOUS HEADS
REQUIRED TO PRODUCE THE RISE OF PRESSURE
GIVEN BY R.C. VENSANO'S FORMULA
$L = 820$ ft.
$V_0 = 2$ ft. per sec.
$a = 4680$ ft. per sec.
$T = 3$ Intervals (=1.05 sec.)
$H_0 = 200'$
$H_0 = 100'$
$H_0 = 50'$
$H_0 = 30'$
Percentage of Gate Opening.
Time, in Intervals

Fig. 8.

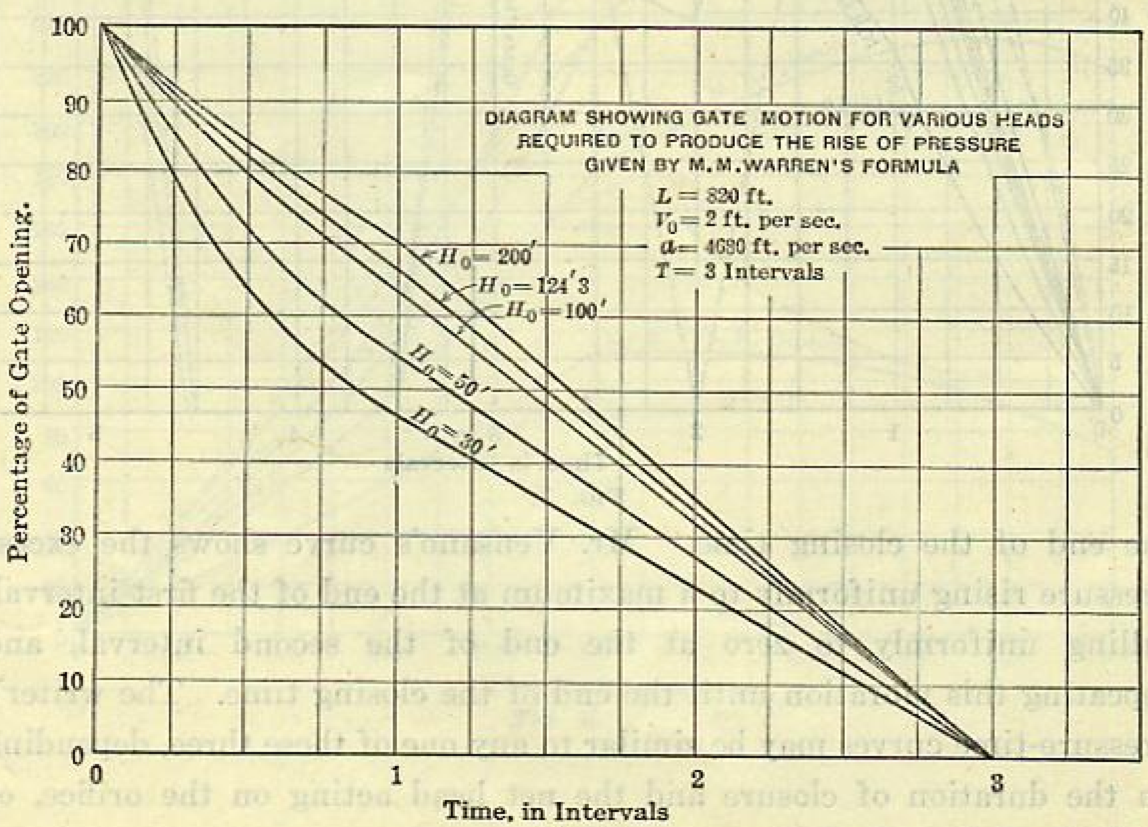


DIAGRAM SHOWING GATE MOTION FOR VARIOUS HEADS
REQUIRED TO PRODUCE THE RISE OF PRESSURE
GIVEN BY M.M. WARREN'S FORMULA
$L = 820$ ft.
$V_0 = 2$ ft. per sec.
$a = 4680$ ft. per sec.
$T = 3$ Intervals
$H_0 = 200'$
$H_0 = 124'3$
$H_0 = 100'$
$H_0 = 50'$
$H_0 = 30'$
Percentage of Gate Opening.
Time, in Intervals

Fig. 9.

Commencing with Fig. 5, it is seen that the pressure-time curve, under the conditions stated, is almost identical with that given by Mr. R. D. Johnson, as far as can be detected by the eye. A careful consideration of the writer's formula, however, will reveal the fact that the pressure-time curve obtained from it is not continuous, but is made up of a series of curves, each being one interval long, and cusps or changes of curvature occur at the end of each interval. Under other conditions, the cusps are plainly visible, as, for example, in Figs. 6 and 7. For any finite conditions, these cusps, theoretically, do not disappear, that is to say, the tangent at the end of one interval never exactly equals the tangent at the beginning of the next interval. Furthermore, the slope at the beginning of the writer's curve is never exactly the same as the slope of the Johnson curve. For all practical purposes, however, the pressure-time curves shown in Fig. 5 (and numerous other curves for similar conditions) are almost identical, and the maximum pressure rise occurs in both of them at the end of the stroke. This is interesting because the two theories from which the respective formulas are derived are totally different. When the conditions are such as to produce practically identical pressure-time curves similar to the foregoing, it indicates that the effect of the compressibility of water, the elasticity of the pipe walls, and the resulting harmonic vibrations in the water column may be neglected, and in such cases the assumptions, on which the Allievi formula has been based, are practically sufficient.

The results obtained from the Warren and Vensano curves are much at variance with the foregoing, as indicated in Fig. 5, and the writer is of the opinion that neither of these formulas can apply for all values of H_0, and, only under certain conditions, which will be mentioned later, are they approximately correct.

As shown in Fig. 3, the maximum pressure rise given by the writer's formulas, as would be expected, is nearly always less than that given by the Allievi formula when the duration of closure is short and when $\frac{H_0}{V}$, the ratio of head to velocity destroyed, is small. When the ratio of head to velocity destroyed is large, the reverse is the case, and, when H_0 becomes infinitely large, the writer's formulas agree with that of Mr. Vensano and give results twice as great as Allievi's formula for any finite duration of closure greater than one interval. Even when the values of maximum pressure rise, as given by the Allievi formula, are in close agreement with those of the writer, both for short and long durations of closure, the respective shapes of the pressure-time curves may be different. Figs. 6 and 7 show clearly the characteristics of the pressure-time curves in a number of typical cases, and a close study of them, together with Fig. 4, will make clear the relation that exists between the rise of pressure and the net head for any velocity destroyed. The shapes of the pressure waves that continue after the gates have been closed, until damped out by friction, are also indicated in Figs. 6 and 7.

Referring now to Fig. 7, it will be noted that the particular case when Mr. Warren's formula gives approximately correct results is when the head is such that the rise of pressure due to gate closure is a minimum for the particular velocity destroyed. This occurs when $H = \frac{3}{7} \frac{a V}{g}$.

Fig. 7 shows, also, the shapes of the pressure-time curves for various heads, and indicates the variation in time when the maximum pressure occurs during closure.

The foregoing examples, except Tables 2 and 5, have all been calculated for uniform gate motion. As the writer's formulas may be applied for any variable rate of gate motion, it is interesting to determine the nature of the gate motion for various heads that would be required to produce diagrams similar to those proposed by Messrs. Warren and Vensano. A diagram of the shape proposed by Mr. Warren may be produced by the simple expedient of increasing the speed of gate travel at the end of the first interval to double the rate of its motion during the first interval and maintaining this double rate until the end of the stroke. Such an operation, though giving the same shape to the pressure-time diagram, will not give, of course, the same rise of pressure as for the same duration of uniform closure, but the latter is given by a gate motion similar to that shown in Fig. 9. If a pressure-time diagram of the same shape as Mr. Vensano's is desired for a low head, a very complicated gate motion would be required, such, for example, as that shown in Fig. 8. In a similar manner, the gate motion required to produce any suggested form of pressure-time curve may be determined.

The manner in which excess pressures vary from gate to origin has been explained in Miss Simin's translation of Professor Joukovsky's paper, although therein worked out on the erroneous assumption of uniform retardation of the velocity of the water flowing in the pipe.

O. V. Kruse, Assoc. M. Am. Soc. C. E., who was associated with the writer at the time this paper was first written, has made an original study of the variation in excess pressure from gate to origin, based on the writer's application of Joukovsky's theory to slow-closing gates. The results of this study, which will appear as a discussion, together with other comments from him, prove that when the duration of uniform closure is less than $\frac{2\,L}{a}$, the maximum pressure is exerted along the pipe to a point where the distance to the origin is equal to $\frac{T\,a}{2}$. From that point to the origin the pressure reduces uniformly to zero. When the duration of uniform closure is equal to, or greater than, $\frac{2\,L}{a}$, the maximum rise of pressure occurs at the gate, and from there to the forebay or origin reduces to zero, uniformly along the length of the pipe.

Fall in Pressure.

Professor Joukovsky's theory of pressure waves applies also to the fall in pressure produced by opening a valve or gate at the end of a pipe line, and the application of his theory to the case when a valve is gradually opened may be made in a similar manner to the foregoing. It is obvious, of course, that the fall in pressure caused by opening a valve in a certain time is not precisely the same as the rise in pressure caused by closing the valve in the same time, because the rate of acceleration cannot be the same as the rate of retardation. It would lengthen this paper unduly, however, to do more than state here that the formulas for fall in pressure may be obtained in a similar manner to those given herein, the principles being the same in both cases.

Experimental Results.

The writer has not yet had an opportunity of testing the formulas presented in this paper by experiments, to show, not only the correctness of the values of maximum pressure rise, but also the shapes of the pressure-time curves under various conditions. It is expected, however, that an opportunity of doing so may possibly occur in the not too distant future. In the meantime this paper is presented in the hope that it will receive criticism or confirmation from those interested in the subject.

The experimental results obtained by Mr. Vensano, and presented in his paper entitled "Pulsations in Pipe Lines", give some striking confirmation of the correctness of the writer's formulas, not only as regards the maximum pressure rise, but also as indicating the resemblance between the calculated and observed pressure-time curves, both before and after the gate was completely closed, and at various points along the pipe line, as well as at the gate.* These, however, were carried out under a very high head (1 260 ft.), and could hardly be accepted as a general proof.

The writer also proposes to show how the pressure-time diagram may be used to determine the velocity of flow in a pipe, for the purpose of measuring the rate of discharge previous to the closing of the gate. This will be made the subject of another paper to be written at a future time.

In conclusion, the writer desires to accord credit to Mr. O. V. Kruse for the valuable help he has given in the preparation of this paper, to Mr. R. L. Hearn for the work of preparing Figs. 8 and 9, and to Mr. R. D. Johnson for the many helpful suggestions he has kindly made.

* See discussion by Norman R. Gibson in *Transactions*, Am. Soc. C. E., Vol. LXXXII, p. 236.

DISCUSSION

Otto V. Kruse,* Assoc. M. Am. Soc. C. E. (by letter).—The writer has studied the phenomena of water-hammer, and offers a short analysis of Miss O. Simin's translation of Prof. Joukovsky's notable work, and an elaboration of certain parts of Mr. Gibson's paper. Mr. Kruse.

Any pipe under pressure and containing water in motion may be assumed as a unit of energy under a condition of equilibrium. If we wish to stop the column of water, it means the application of a certain amount of force during a certain time to bring about the conversion of energy and restore the equilibrium. In analyzing the phenomena which take place, the writer believes that the most important point to study is the rate of destruction of velocity as affected by the net head, because this determines to a large extent the rate of pressure rise and hence the maximum pressure rise. Apparently, Allievi used this as a basis for the development of his formula, and this, of course, applies to Johnson's formula, which is the same as Allievi's. Any formulas, such as Warren's and Vensano's, which do not take this into account must be incorrect, except for one set of conditions. The curves given by Mr. Gibson show how far from the truth they may be.

In accordance with the ordinary methods of integration, it may be considered that the gate movement throughout the duration of the stroke is made up of an infinite number of small instantaneous movements. In the case of the formulas of Allievi and Johnson, which involve the dynamic forces, each little movement of the gate produces a pressure:

$$h = \frac{L\,dv}{g\,dt}$$

The conditions existing, then, at the end of any small movement are:

1. The original velocity has been decreased by the amount of the velocity destroyed during this first movement;
2. A dynamic pressure has been created, due to the destruction of velocity.

A new velocity now exists, which is dependent, not on the original net head, but on a new head made up of the original net head and the dynamic head. As brought out by Mr. Johnson, these curves of dynamic pressure rise and decrease in velocity may be calculated by a simple method of arithmetic integration, using only the relation, $h = \frac{L\,dv}{g\,dt}$. The Allievi or Johnson formula will give a figure for maximum pressure rise, and Mr. Johnson has also developed a formula for the shape

* Philadelphia, Pa.

Mr. Kruse. of the pressure curve, these formulas giving the same results as can be obtained approximately by arithmetic integration.

The question to be considered now is: To what extent, and how, do the compressibility of water and extension of the pipe walls affect the pressure and velocity curves? We know that, as the time of closing grows shorter, the Allievi formula approaches infinity. On the other hand, Joukovsky proved that the maximum pressure which obtained for instantaneous closing depended on the velocity destroyed and the speed of propagation of the pressure wave, and could not exceed "maximum water-hammer." It is not difficult to see that, for a long closing time, the compressibility of the water and extension of the pipe walls have very little effect on the maximum pressure, but, as the time of closing grows shorter, the properties of the materials have a greater and greater effect. The writer believes that Mr. Gibson's formulas have successfully supplied this missing gap between Allievi's and Joukovsky's theories, and serve to give correct results for all conditions.

Professor Joukovsky made a great many experiments to determine the magnitude of maximum water-hammer and the speed of propagation of the pressure wave, all based on instantaneous closing of the gates. The details of the experiments will not be given here, but the following is a synopsis of the theory, taken verbatim from Miss Simin's translation:

"In Fig. [10] let *A B* be a pipe, in which water flows with velocity, *v*, from the origin *A*, past the gate, *B*. If, now, the flow is suddenly stopped by a rapid shutting of the gate, *B*, the kinetic energy of the water column, *A B*, will cause an increase of pressure in the pipe.

"Let us consider the column of water, *A B*, as divided into *n* very small equal sections, 1, 2, 3,....(n — 1) and *n*.

"The phenomena of water-hammer take place in a series of cycles, each consisting of four processes, as follows:

"(1) Section 1, meeting, in the gate, an obstacle to its movement, will be compressed and will stretch the pipe wall surrounding it. All the kinetic energy of this section of water will be used up (*a*) in its own compression, resulting in the increase of pressure by an increment, *P*, and (*b*) in the corresponding stretching of the walls in section 1 of the pipe. As a result of this action, section 1 of the water column has left vacant behind itself a small space, to be occupied by a part of the next arriving section 2. Consequently it is only after section 1 has been stopped and compressed, and after the small space thus left has been filled, that section 2 can be arrested and compressed.

"Now the kinetic energy of section 2 must be expended in some way. Will it increase the pressure upon the gate, which has already been caused by the arrest of section 1? No, and for the following reason:

"The pressure upon the gate depends entirely upon the pressure, *P*, sustained by section 1, which is now in static condition.

"The pressure upon the gate could therefore be increased only if section 1 could be further compressed, and this could take place only if the pressure upon the surface between it and section 2 (which we may imagine to be a thin piston) could be greater from the side of section 2 than it is from the side of section 1; and this is impossible, because section 2 has only the same kinetic energy as section 1, and this energy will (as in the case of section 1) be used up entirely in compressing the water of the section (section 2) only to the same additional pressure, P, and in stretching that part of the walls surrounding section 2. Mr. Kruse.

"The same is true of each following section, 3, 4,.... $(n - 1)$ and n; each of these sections, as it is arrested, being compressed to the pressure, P.

"During process (1) a small quantity of water flows from the reservoir into the pipe, to occupy the space formed by the compression of the water and the extension of the pipe walls.

"Finally, when all the sections have been arrested, the entire column will be under the pressure, P. The entire energy of the water column is now stored (as potential energy) in elastic deformation, viz., in the compression of the water column and in the extension of the pipe walls.

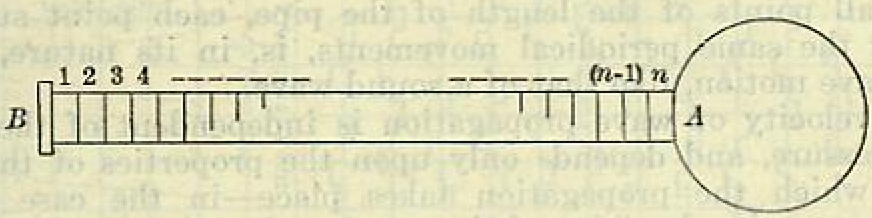


FIG. 10.

"But this condition cannot be maintained; for

"(2) As soon as the additional pressure, P, has been produced in the last section, n, the water in that section will again expand, and the walls of that section of the pipe will again contract, restoring the original conditions in that section, and pushing the water of that section back into the reservoir from which the pipe issues, and restoring the original normal pressure in section n.

"This operation will now be repeated by each section $(n - 1)$,....4, 3, etc., in turn, until all the potential energy, stored in the water column when it was under the pressure, P (neglecting the portion lost in friction), has been reconverted into kinetic energy.

"During process (2) the water which entered the pipe during process (1) is forced back into the reservoir.

"The condition of the water column is now what it was just before the gate was closed, except that its velocity, v, has now the opposite direction, *i. e.*, toward the origin.

"(3) The kinetic energy of the water column, moving toward the origin or away from the gate, is now reconverted into potential energy, which manifests itself in an extension of volume of the water to a subnormal pressure beginning with section 1, and concluding only when the entire water column has been reduced to the subnormal pressure.

"During process (3) water continues flowing from the pipe into the reservoir.

"(4) When the subnormal pressure has been established throughout the length of the pipe, and all the water has come to rest, the water from the reservoir will again direct itself into the pipe, restoring

Mr. Kruse. the normal pressure, first in section n, next to the reservoir, and then, in rapid succession, in the other sections $(n - 1), \ldots 4, 3$, etc., until, when the normal pressure reaches the gate, we have once more the conditions which existed just before the gate was closed, viz., the normal pressure is restored and the water is moving toward the gate with the original velocity, v.

"We have now followed these pulsations of pressure (with the accompanying transformations of energy and flow of water into and back from the pipe) through a complete cycle of four movements, each extending through the length of the pipe. For convenience, we may consider two successive movements of this kind as a 'round trip' through the pipe.

"The gate remaining closed, the whole process is now repeated in a second cycle, which, in turn, is followed by a third, and so on, the amplitude of the pressure vibrations gradually diminishing (because of friction) until the pipe and the water come to a state of rest.

"But although the intensity of the pressure becomes gradually less, the time required for each cycle remains constant for all repetitions.

"This propagation of pressure, consisting of its transmission through all points of the length of the pipe, each point successively repeating the same periodical movements, is, in its nature, simply a case of wave motion, like that of a sound wave.

"The velocity of wave propagation is independent of the intensity of the pressure, and depends only upon the properties of the medium through which the propagation takes place—in the case of water-hammer, upon the elasticity of the water and of the pipe."

Mr. Gibson has applied this theory to slow-closing gates by considering that each infinitesimal movement of the gate is instantaneous and produces a small rise of pressure which travels through the penstock in the same manner that a wave produced by total instantaneous closing would travel. These small gate movements occur in succession and produce in turn their pressure waves which travel back and forth through the penstock. An algebraic sum of these waves gives the rise of pressure existing at any time during the closing stroke.

The writer is of the opinion that the pressure existing throughout the length of the pipe for a slow-closing gate varies almost directly from a maximum at the gate to zero at the point of relief, except where the time of closing is less than $\frac{2L}{a}$.

In Fig. 11, the pressure existing at four points on the penstock has been plotted. The magnitude of the individual pressure waves is not to scale, but their values are shown for the particular example taken. The pressure existing at a point $\frac{L}{2}$ from the gate is approximately one-half the pressure existing at the gate. The pressure existing at a point $\frac{3L}{4}$ from the gate is approximately one-quarter of the pressure existi[illegible] at the gate.

Mr. Kruse.

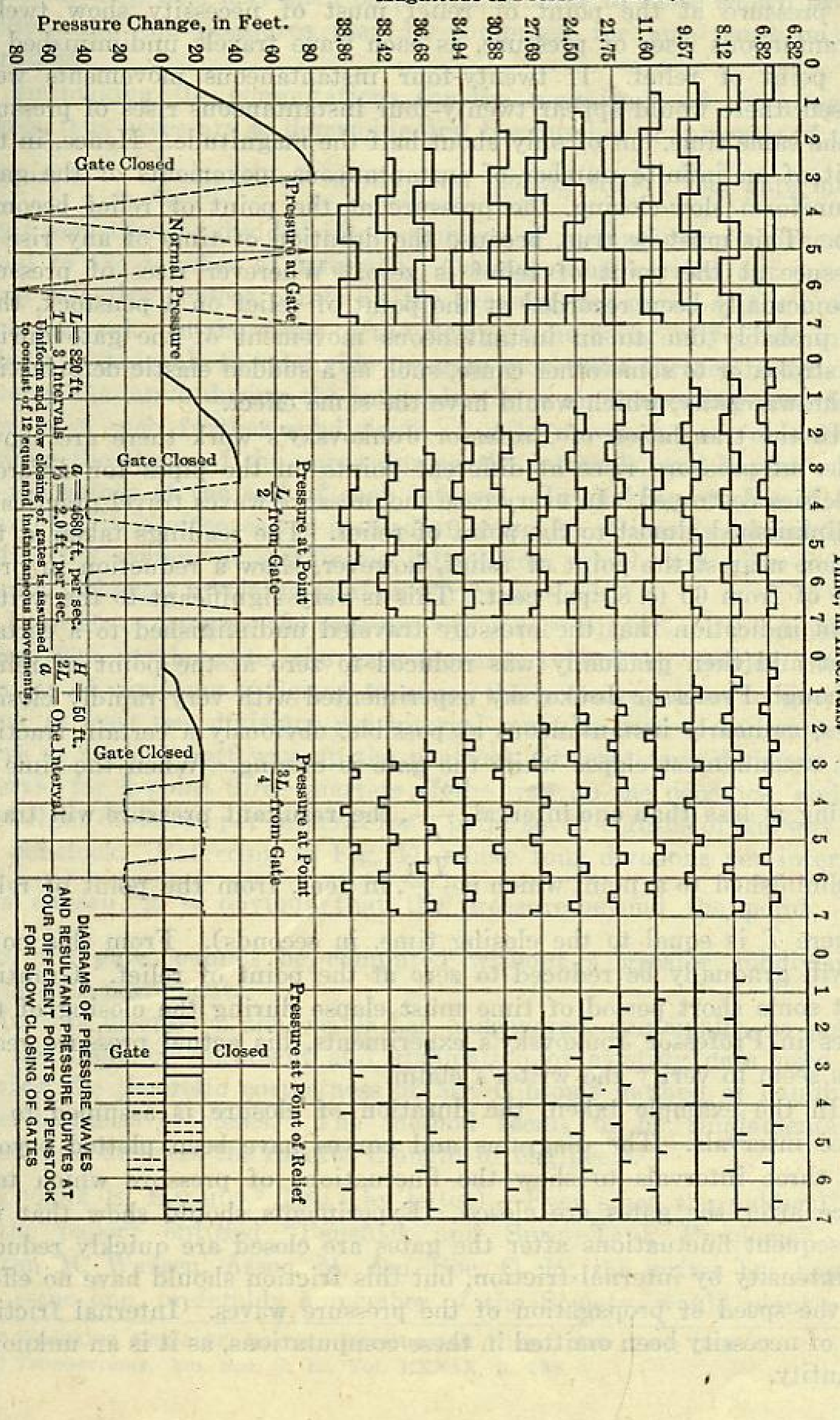


FIG. 11.

DIAGRAMS OF PRESSURE WAVES AND RESULTANT PRESSURE CURVES AT FOUR DIFFERENT POINTS ON PENSTOCK FOR SLOW CLOSING OF GATES

Mr. Kruse. The diagram showing the pressure existing at the point of relief consists of a series of instantaneous rises above normal. On account of the assumption of twelve instantaneous movements of the gate, the pressure at the point of relief must of necessity show twelve instantaneous rises of pressure, as each wave travels undiminished to the point of relief. If twenty-four instantaneous movements were chosen, there would appear twenty-four instantaneous rises of pressure in the same time, but of only about half the magnitude. Hence, in the limit of an infinite number of instantaneous movements of the gate, or uniform slow-closing, the pressure at the point of relief becomes zero. This must be true, because the duration of time of any rise of pressure at the point of relief is zero. Wherever rises of pressure have actually been recorded at the point of relief on a penstock, they are probably due to an instantaneous movement of the gate during the stroke, or to some other cause, such as a sudden elastic deformation of the waterway, which would have the same effect.

In the translation of Professor Joukovsky's work there are shown tables of pressure rises at different points on the pipes for different velocities destroyed. In every case the pressure waves travel practically undiminished almost to the point of relief. The readings taken at the station nearest the point of relief, however, show a reduction in pressure of from 60 to 80 per cent. This is very significant to the writer, as an indication that the pressure traveled undiminished to a certain point and then gradually was reduced to zero at the point of relief. Although Professor Joukovsky experimented with very rapidly closing gates, as nearly instantaneous as possible, obviously a certain fraction of a second must elapse while the gate is closing. When the time of closing is less than one interval, $\frac{2\,L}{a}$, the resultant pressure will travel undiminished to a point which is $\frac{T\,a}{2}$, in feet, from the point of relief (where T is equal to the closing time, in seconds). From this point it will gradually be reduced to zero at the point of relief. Granting that some short period of time must elapse during the closing of the gates in Professor Joukovsky's experiments, his actual pressure readings seem to verify the writer's claim.

In the example taken, the duration of closure is assumed to be three intervals. The diagrams and curves have been plotted beyond the three intervals to show the fluctuations of pressure which take place after the gates are closed. Experiments should show that the subsequent fluctuations after the gates are closed are quickly reduced in intensity by internal friction, but this friction should have no effect on the speed of propagation of the pressure waves. Internal friction has of necessity been omitted in these computations, as it is an unknown quantity.

Any other example than that shown by Fig. 11 may be chosen, and the pressure waves may be computed by Mr. Gibson's formulas. A series of diagrams may then be plotted, similar to those shown by the writer, and the pressures may be computed for any point on the penstock. Mr. Kruse.

In making the computations for the magnitude of the pressure waves, it is necessary to determine first the number of intervals of time $\left(\text{equal to } \frac{2\,L}{a}\right)$ contained in the total closing time. The gate movement may then be divided into a number of instantaneous movements equal to the number of intervals. Computations carried out on this basis will give correct points on the pressure-rise curve at the end of each interval. In the event of the closing time being only a few intervals, it is desirable to subdivide the gate movement further in order to develop the curve during the intervals. It is an interesting fact, however, that such further subdivision of gate movement serves only to develop the shape of the curve during the intervals, and has no effect on the values at the end of each interval. In making studies of this character, it is often easier instead of using Mr. Gibson's formulas, to utilize the system of arithmetic integration, which will give correct results, although involving trial-and-error methods.

In order to compute the pressure-rise curve at points on the penstock other than at the gate, the gate movement must be divided into a number of divisions sufficient to cause the pressure waves to overlap. Thus, at least two divisions per interval must be used to compute the curve at a point half way up the penstock, at least four divisions per interval for a point three-quarters of the way up the penstock, and at least eight divisions per interval for a point seven-eighths of the way up the penstock. Referring to Fig. 11, where four divisions per interval were chosen, it is obvious that the pressure beyond the point, $\frac{3\,L}{4}$ from the gate, cannot be computed without a greater subdivision of the gate movement.

In this discussion the writer has endeavored, principally, to point out the trend of argument and application of existing data used in proving the theoretic correctness of Mr. Gibson's method of handling water-hammer problems. The method seems to be fundamentally sound, and deserves the careful thought of engineers.

Eugene E. Halmos,* Esq. (by letter).—Ever since the publication of the paper† entitled "Penstock and Surge-Tank Problems" by Minton M. Warren, Assoc. M. Am. Soc. C. E., the writer has hoped that some one, preferably a member of the Society, would point out Mr. Halmos.

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† *Transactions*, Am. Soc. C. E., Vol. LXXIX, p. 238.

Mr. Halmos. the injustice done to one of the greatest hydraulic engineers of the present age by the form of, and the comments appended to, Mr. Warren's presentation of what he termed "Alliévi's formula."

Unfortunately, Mr. Warren has not been corrected. This formula was accepted by American engineers as representing Alliévi's solution of the problem of water-hammer in pipe lines, and, consequently, it has been severely criticized and misinterpreted.

The value of Mr. Gibson's excellent paper is also seriously impaired by the references, both in the text and in the diagrams, to "Alliévi's formula" as taken from Mr. Warren's paper, and the writer would suggest that, after perusal of the following, Mr. Gibson omit all reference to Alliévi, except to state that his solution agrees, in every particular, with that of Alliévi.

The writer proposes to submit a brief summary of Mr. Alliévi's formulas, which, in his judgment, represent a complete solution of water-hammer problems as regards determination of the pressure at the gate or at any other section of the conduit, at any instant, and under any conditions of closure and physical characteristics of the plant. It is more general than Mr. Gibson's solution, inasmuch as it also includes the determination of the pressure variations after the stopping of the gate movement, and as it gives a method of estimating the interval in which the pressure maximum will occur.

As the present paper and the previous ones deal almost exclusively with the case of water-hammer during the closure of gates, and with the variation of the pressure at or near the gates, the writer will only quote Alliévi with regard to formulas applicable to such cases. The writer, however, intends to translate Alliévi's book into English, which, it is hoped, will be of great help to designers of hydraulic plants and apparatus.

In the following, the formulas representing the relation of time and pressure are given for three phases, namely:

the first phase, for which $t \leqq \frac{2L}{a}$;

the second phase, for values of t beginning with $t = \frac{2L}{a}$ and ending at T;

and the third phase, $t > T$.

Notation.—

a = velocity of propagation of pressure along pipe;

c_0 = uniform velocity of water in pipe before beginning of gate closure;

C = variable velocity of water in pipe at any instant during gate closure;

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u_0 = uniform velocity of water through gate orifice before beginning of closure;

u = variable velocity of water through gate orifice during gate closure;

t = time, in seconds, recorded from beginning of gate movement;

T = duration of gate movement, assumed to be continuous;

L = length of pipe conduit;

y_0 = pressure head on pipe line at gate before closing begins;

y = variable head at gate due to closure.

A.—General Formulas.

(1) *Variation of Pressure Head During and After the Closing Motion of the Gates.*—

First phase, $t \leqq \frac{2L}{a}$.

For this interval, the pressure head, y, at any instant can be computed from the equation

$$y^2 - 2y\left(H + \frac{a^2 \psi^2(t)}{g}\right) + H^2 = 0 \ldots\ldots\ldots\ldots (1)$$

in which $H = y_0 + \frac{a\, c_0}{g}$

and $\psi(t)$ is the ratio of the instantaneous area of discharge to the (assumed) constant area of the penstock, that is, $\psi(t) = \left(1 - \frac{t}{T}\right) \frac{c_0}{\sqrt{2\, g\, y_0}}$ the value of $\psi(t)$ being taken at the instant for which the pressure head, y, is calculated.

It will be noted that if the gate is closed in a time, $T \leqq \frac{2L}{a}$, $\psi(T) = 0$, and the equation reduces to

$$y^2 - 2Hy + H^2 = 0$$

$$y = H = y_0 + \frac{a\, c_0}{g} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

which shows that in such a case the maximum pressure head is independent of the actual length of the time of closure.

Of the two roots of Equation (1) one is greater, the other is smaller, than H. This latter is the true value of the pressure head at any instant.

Second phase, $\frac{2L}{a} \leqq t \leqq T$.

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During this period of the gate movement, the following equation applies for the determination of the pressure head at any instant:

$$y^2 - 2\,y\,\left(H - 2\,f + \frac{a^2\,\psi^2\,(t)}{g}\right) + (H - 2\,f)^2 = 0 \ldots\ldots (3)$$

Of the two roots of Equation (3), that giving the smaller value should be used.

It will be noted that this equation differs from Equation (1) only in so far that, instead of H in the latter, $(H - 2\,f)$ appears in Equation (3).

The values of the function, f, should be determined in the following manner:

For the interval, $0 < t < \frac{2\,L}{a}$, $y - y_0 = F\,(t)$, y being calculated from Equation (1).

The velocity, C, in the conduit (at the gate) can be computed from the equation

$$y + \frac{a}{g}\,C = H$$

for this interval.

For the interval, $\frac{2\,L}{a} < t \leqq \frac{4\,L}{a}$, $f = F\left(t - \frac{2\,L}{a}\right)$; in other words, the values of $F\,(t)$ found in the first interval should be substituted for f in Equation (3) to obtain the value of y for every subdivision of the second interval. Then, the corresponding values of the velocity, C, may be found by $y + \frac{a}{g}\,C = H - 2\,f$, and the value of $F\,(t)$ for the second interval by $F\,(t) = \frac{a}{g}\,(c_0 - C) - f$.

For the third interval, $\frac{4\,L}{a} < t \leqq \frac{6\,L}{a}$, the values of $F\,(t)$ so found for the subdivisions of the second interval should be substituted for f in Equation (3) to get y, and so on, for all subsequent intervals until the gate has stopped moving or has become entirely closed.

Third phase, $t > T$.

If the movement of the gate is stopped at an instant such that $t_1 < \frac{2\,L}{a}$, the pressure head, y, remains constant until $t = \frac{2\,L}{a}$. From this instant on, or from the time of the stoppage, in the case, $t_1 > \frac{2\,L}{a}$, there is developed a hydrodynamic phenomenon in the form of an asymptotical approach of the pressure head, y, to the new constant head, y_0, of uniform flow. There are three different cases.

First.—If $a\ \psi\ (t_1) > \frac{u_0 + u_1}{2}$, in which case y approaches y_0 without oscillations. Mr. Halmos.

Second.—If $a\ \psi\ (t_1) < \frac{u_0 + u_1}{2}$, in which case y approaches y_0 through oscillations of diminishing amplitude.

Third.—If the gate is entirely closed, $\psi\ (t_1) = 0$, the pressure head (disregarding the dampening effect of hydrodynamic friction) oscillates indefinitely, with a constant amplitude, between the value of y, which occurs at the instant of closing, and $2\ y_0 - y_1$, the amplitude being $y_1 - y_0$.

(2) *Estimating the Phase in Which the Maximum Pressure Will Occur.*—Assuming a linear closure of the gates:

(a) If $a\ c_0 < 2\ g\ y_0$, then the pressure head at the end of the first phase $\left(t = \frac{2\ L}{a}\right)$ will always be greater than the average maximum pressure (found by Equation (4)) occurring during the second phase.

(b) If $2\ g\ y_0 < a\ c_0 < 3\ g\ y_0$, the maximum pressure head occurs either at the end of the first phase, or during the second phase, according to whether the closing time, T, is smaller or greater than

$$\frac{a\ c_0 - g\ y_0}{a\ c_0 - 2\ g\ y_0} \cdot \frac{L}{a}.$$

(c) If $a\ c_0 > 3\ g\ y$, then the maximum pressure will always occur during the second phase and will be greater than that at the end of the first phase.

Equations (1), (2), and (3) are the general formulas derived by Alliévi, by the help of which complete pressure-time curves can be worked out for any assumed conditions of T, L, a, c_0, and y_0. The writer is very much pleased to state, that, by the use of these equations, he was able to check accurately all the curves presented by Mr. Gibson. He wishes to compliment the author in having found, quite independently, and by a simpler mathematical method than Alliévi's functional derivation, the correct formulas for the representation of pressure-time curves and related phenomena.

B.—Special Formulas.

It has been found by experiment, and by working out pressure-time curves for a great number of cases actually met in practice, that, on the assumption of linear gate closure:

(d) The pressure behind the gate remains practically constant for $\frac{2\ L}{a} < t \leq T$ (y is not a function of time or $\frac{d\ y}{d\ t} = 0$ between

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these limits) and, therefore, for this period, the pressure is independent of the elastic qualities of the water and the conduit, in other words, independent of (a);

(e) The pressure is distributed in a linear way along the pipe ;

(f) The velocity of the water, C, is the same at any instant at any section of the conduit $\left(\frac{d\,c}{d\,x} = 0\right)$.

Taking into consideration the observations noted under (d), (e), and (f), a good average value of y maximum during the second phase can be obtained by solving the equation

$$Z^2 - Z\,(n^2 + 2) + 1 = 0 \quad \text{.................} \quad (4)$$

where $$Z = \frac{y_{max.}}{y_0} \text{ and } n = \frac{c_0\,L}{g\,T\,y_0}.$$

This equation is identical with Equation (C) in Mr. Warren's paper. In order that it should be valid, it is necessary:

(1) That T be greater than $\frac{2\,L}{a}$;

(2) That the gate movement (the rate of reduction of the area of the outlet) be a linear function of the time;

(3) That, in case of partial closure, that value of T should be used which would be obtained if the gate should be completely closed at the same rate of speed;

(4) That the physical data, as L, a, y_0, and c_0, shall conform to those ordinarily met in practice.

In order that Equation (4) should produce the maximum pressure head obtaining during the whole movement of the gate, it is necessary that $a\,c_0$ shall take the values defined under (b) and (c).

It is evident that Mr. Warren committed a serious error by omitting the publication of Alliévi's general Equations (1), (2), and (3), and by omitting the proper definition of Equation (4) and the limitations of its applicability as defined by Alliévi.

The only ambiguity which may occur in using this formula is the lack of a better definition of Condition (4). This is extremely difficult, on account of the great number of variables entering into the problem. The writer, however, proposes, without being able to give strict analytical proof at the present time, that Equation (4) should be used only, if

(α) $$\frac{L\,c_0}{g\,T\,y_0} < \frac{1}{2}\sqrt{\frac{a\,c_0}{y_0\,g} + \frac{y_0}{y_0 + \frac{a\,c_0}{g}} - 1}; \text{ and,}$$

(β) $a\,c_0 > 2\,g\,y_0$.

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As L, c_0, a, and y_0 are dependent on rather unalterable physical conditions, if Condition (β) is satisfied, Condition (α) can be fulfilled by determining T according to the proposed formula.

If, besides Conditions (α) and (β), Conditions (b), (c), (1), and (2) are also fulfilled, Equation (4) will give the absolute maximum head at the gate during the movement of the apparatus.

It will be found that,

$$\text{for } c_0 \geqq 10 \text{ ft. per sec.}$$

$$y_0 = \text{between 100 ft. and 500 ft.}$$

$$T \geqq 6 \text{ intervals,}$$

all the foregoing conditions are fulfilled, which means that, in most practical cases, Equation (4) actually furnishes the maximum pressure head at the gate.

If the elasticity of the water and the conduit is neglected for the whole time of the movement of the gate, then, for the conditions for which Equation (4) is applicable, the pressure-time curve can be platted from the equation:

$$\log. \frac{\left(\sqrt{u_0^2 + u_R^2} + u - u_R\right)\left(\sqrt{u_0^2 + u_R^2} - u_0 + u_R\right)}{\left(\sqrt{u_0^2 + u_R^2} - u + u_R\right)\left(\sqrt{u_0^2 + u_R^2} + u_0 - u_R\right)} = \frac{\sqrt{u_0^2 + u_R^2}}{u_R} \log. \frac{T}{T - t} \quad \ldots\ldots\ldots\ldots\ldots (5)$$

in which $u_R = K \dfrac{L}{T}$, where K is the ratio of the area of the gate opening at $t = 0$ to the area of the pipe, and

$$u_0 = \sqrt{2\, g\, y_0},\quad u = \sqrt{2\, g\, y}.$$

This curve will have a maximum at $t = T$ if Conditions (b) and (c) are fulfilled, and will have a practically horizontal position in the second phase if Condition (α) is observed. It will differ very little from the true curve of pressure variations, and is applicable to practically the whole range of actual water-power problems.

It will be noted that Equations (4) and (5) are identical with Equations (6) and (7) given by Mr. R. D. Johnson in his discussion* of Mr. Warren's paper. Great credit is due to Mr. Johnson for having grasped correctly the essential features of this difficult subject, and for his wonderful mathematical skill in solving by independent methods the intricate problem of water-hammer.

In conclusion, the writer wishes to observe, that it would be desirable to encourage the use of Alliévi's, Gibson's, and Johnson's formulas for

* *Transactions*, Am. Soc. C. E., Vol. LXXIX, p. 280.

Mr. Halmos. water-hammer, and to reject and discourage the use of other formulas, some of which, such as Mr. Warren's, are never true, not even accidentally.

Mr. Johnson. R. D. JOHNSON,* ESQ. (by letter).—Mr. Gibson's remarkable paper deserves the closest attention on the part of all who are interested in the water-hammer problem.

He remarks that the theory has not been experimentally confirmed, but it is so obviously complete and correct as scarcely to need such confirmation to establish strong faith in its accuracy.

The writer has been especially interested in a completely correct treatment of this subject, in order to be able to fix the limits of K within which his pressure-time curve, previously presented to the Society, is sufficiently accurate for practical purposes.

It is found, fortunately, that this smooth curve,† $\log. x = n \log. \frac{T}{T - t}$, is applicable with much precision to such a wide range of conditions as to justify its use for practically all the water-hammer problems arising in connection with the design and operation, under ordinary heads, of water turbines, when properly regulated.

By differentiating Mr. Gibson's equations, it is possible to express rigidly correct values of the tangents to the true pressure-time curve at (a), the beginning of gate closure, (b), the end of the first interval, and (c), the beginning of the second interval.

Using Mr. Gibson's nomenclature, with omission of subscripts, that is, putting $R_0 = R$, and $S_{T_1} = S$, and $H_0 = H$, these tangents may be expressed as follows:

$$\tan. (a) = \frac{2\,R\,H}{(R + 2\,H)\,T}$$

$$\tan. (b) = \frac{R^2\,(T - T_1)}{H\,T^2}\left\{\frac{S + 2\,R + 2\,H}{\sqrt{S\,(S + 4\,R + 4\,H)}} - 1\right\}$$

$$\tan. (c) = \frac{R^2\,(T - T_1)}{H\,T^2}\left\{\frac{S\,(Z + 1) + 2\,R + 2\,H}{\sqrt{S\,(S + 4\,R + 4\,H)}} - (Z + 1)\right\}$$

$$\text{where } Z = \frac{4\,H^2\,T}{R\,(R + 2\,H)\,(T - T_1)}.$$

It is clear that the effects of vibration in the water column, due to elasticity, tend to disappear when the conditions are such that tan. (b) and tan. (c) approach equality; also, if the change in the rate of pressure rise at the beginning of the second interval, due to the effect of

* New York City.

† The derivation of this curve, Equation (7), is given in *Transactions*, Am. Soc. C. E., Vol. LXXIX, p. 280.

Mr. Johnson.

the returning wave, is less than the rate at which the pressure begins to rise when the gate starts to move, it is evident that the conditions are such that the effect of elasticity is rapidly dying out, and that it should be scarcely appreciable during the remainder of the closing period.

If, then, tan. (b) — tan. (c) $<$ tan. (a), the writer's Equation (7), above referred to, or the maximum value of h, given by the so-called Alliévi formula, may be used with much precision.

This may be shown to be the case when,

$$K > \frac{2}{\sqrt{3\,r^2 + 1} - 1} \qquad (1)$$

where r is the ratio, $\frac{T - T_1}{T}$; and $K = \frac{R}{H}$.

Formula (1) marks the lower limit for values of K, below which it is not safe to use Equation (7). To find the upper limit of safety, for values of K, it is noted that, as H decreases, the pressure-time diagram passes through a transition stage where an ogee form of curve most closely approaches to a straight line, and, that thereafter, as H continues to decrease, Equation (7) begins to produce a curve sharply upturning and curving one way, for the most part, as its extremity reaches toward infinity, thus leading to serious errors in its results.

By trial, assisted by arithmetic integration, it has been noted that, for large values of n, the head may be reduced to about one-half the value which produces the approximate straight-line diagram.

To find this point we may equate the value of h_{max} in the so-called Alliévi formula to the value of the pressure rise found at the end of a straight-line pressure-time diagram, enclosing the same impulse area, and solve for the value of H.

In this way it is found that

$$K = 3n \qquad (2)$$

(where n is the closing time expressed in intervals), and Equation (7) may be used so long as

$$K = \text{or} < \frac{6\,n^2}{n + 1} \qquad (3)$$

which, for large values of n, is nearly twice as great as the value given by Formula (2), but which becomes equal to 3, as it necessarily must, when n becomes unity.

Formula (3) marks the upper limit for values of K, above which it is not safe to use Equation (7). Therefore, the smooth logarithmic curve given by Equation (7) may be applied only when K lies between the values given in Formulas (1) and (3), and preferably, also, when n is not less than three intervals.

Mr. Johnson.

Fig. 12 shows that Equation (7) covers a very wide range of conditions, and, furthermore, in the design of a water-turbine installation, the values of V and n should, when possible, be adjusted so that Equation (7) will be correctly applicable; and particular attention should be given to the limitations set forth in Formulas (1) and (3), in order that the speed of the water unit may be regulated most effectively.

The expressions for the tangents, (a), (b), and (c), are rigidly accurate, and form a substantial basis for further possible study; for example, if tan. (c) be equated to zero, the limiting value of n may be found when the maximum pressure rise occurs at the end of the first interval. This is thus found to be the case, always, when

$$n > \frac{4\,K^2 + 8\,K}{12 - K^2 - K^3} \quad \text{.......................(4)}$$

It is seen that this expression is positive and finite only so long as $K < 2$; therefore, K must always be less than 2, in order that the maximum rise shall occur at this point. It does not follow that this will be the case merely because $K < 2$, unless Formula (4) also applies.

In support of Mr. Gibson's work by experimental observation, it may be pointed out that much information is available for cases which occur within the limits here set forth, and, inasmuch as his complete work incidentally includes such cases, it may safely be predicted that no error in theory will be discovered by experiment.

The writer, himself, has confirmed Equation (7) a great many times, under a variety of conditions, in the past twenty years, and, because of such confirmation, he was not led, from practical necessity, to a more thorough investigation of the theory.

In conclusion, the writer wishes to urge those interested to a further study of the limits here graphically presented, with the idea either of confirming or improving them, because it is not profitable to resort to the very tedious processes incident to the methods presented by Mr. Gibson, except when a comparatively simple formula is inapplicable. The chart, Fig. 12, indicates that this is very rarely the case, and practically never need be, with proper care in the design and operation of any water system.

It is interesting to note that Mr. Gibson's methods may be used to work out a system of perfectly general cases, thus enlarging, indefinitely, the scope of such diagrams as he has presented, which are based, merely, on a specially selected set of data.

This is made possible by simply substituting in his scale of ordinates, values of the ratio $(h_{max.}) \div H$ in place of $(h_{max.})$, and substituting for the values of H, written on the curves, the values of K.

This trifling change, it will readily be seen, will make such diagrams applicable to an infinite variety of special cases.

In this manner, the chart, Fig. 13, has been roughly prepared, and, for the range of values covered, it is not necessary to resort to formulas. Mr. Johnson

All combinations of a, V, and H which produce a constant value of K must invariably result in a definite single value of the ratio

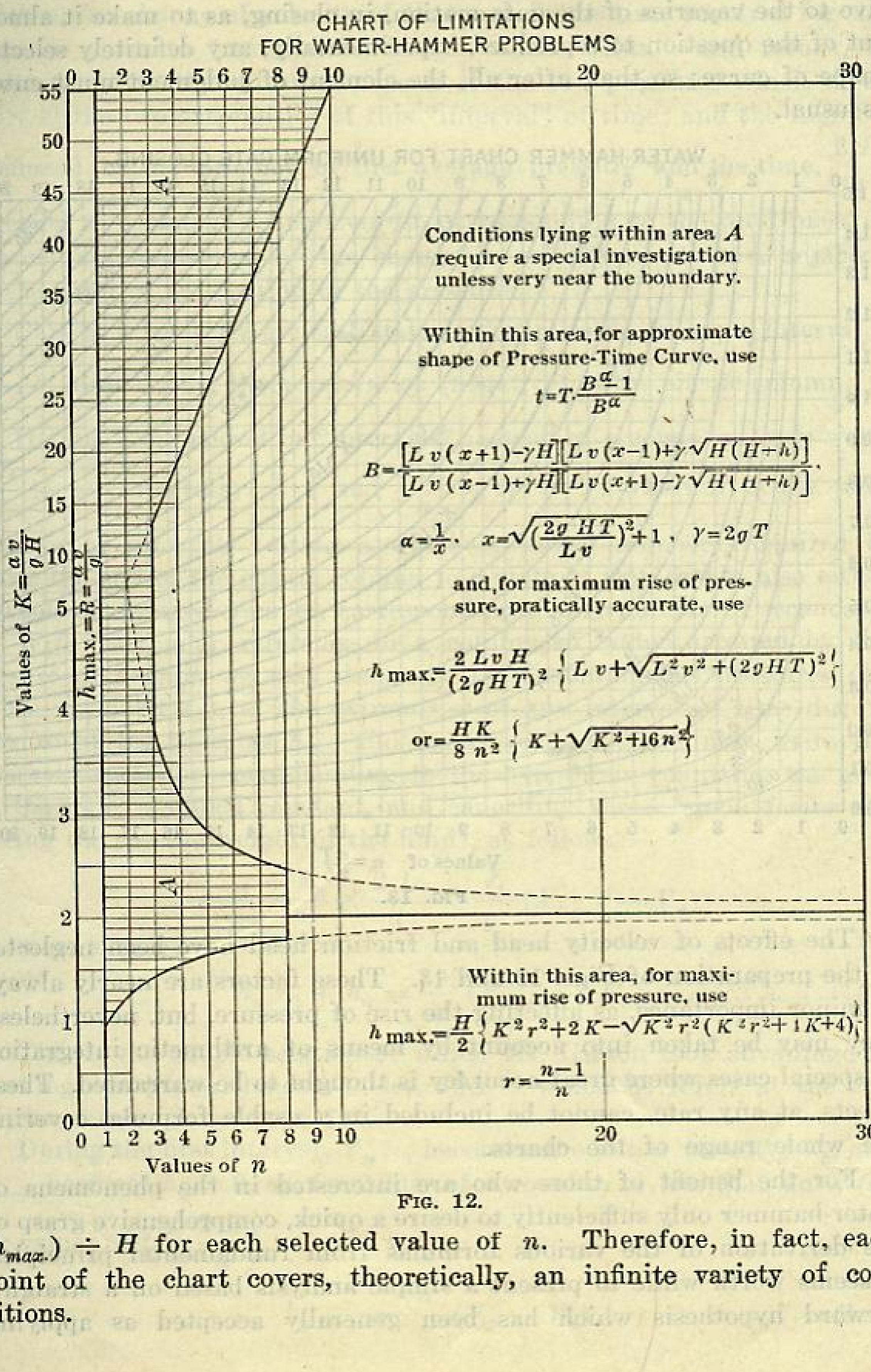

FIG. 12.

$(h_{max.}) \div H$ for each selected value of n. Therefore, in fact, each point of the chart covers, theoretically, an infinite variety of conditions.

Mr. Johnson.

In the use of the chart or simple formulas to replace the accurate but tedious methods, it should be borne in mind that it is not ordinarily practically necessary that the agreement should be perfect, because, as Mr. Gibson has shown, the shape of the curve is so sensitive to the vagaries of the gate motion, in closing, as to make it almost out of the question to reproduce, experimentally, any definitely selected shape of curve; so that, after all, the element of judgment must enter, as usual.

WATER-HAMMER CHART FOR UNIFORM GATE CLOSING.

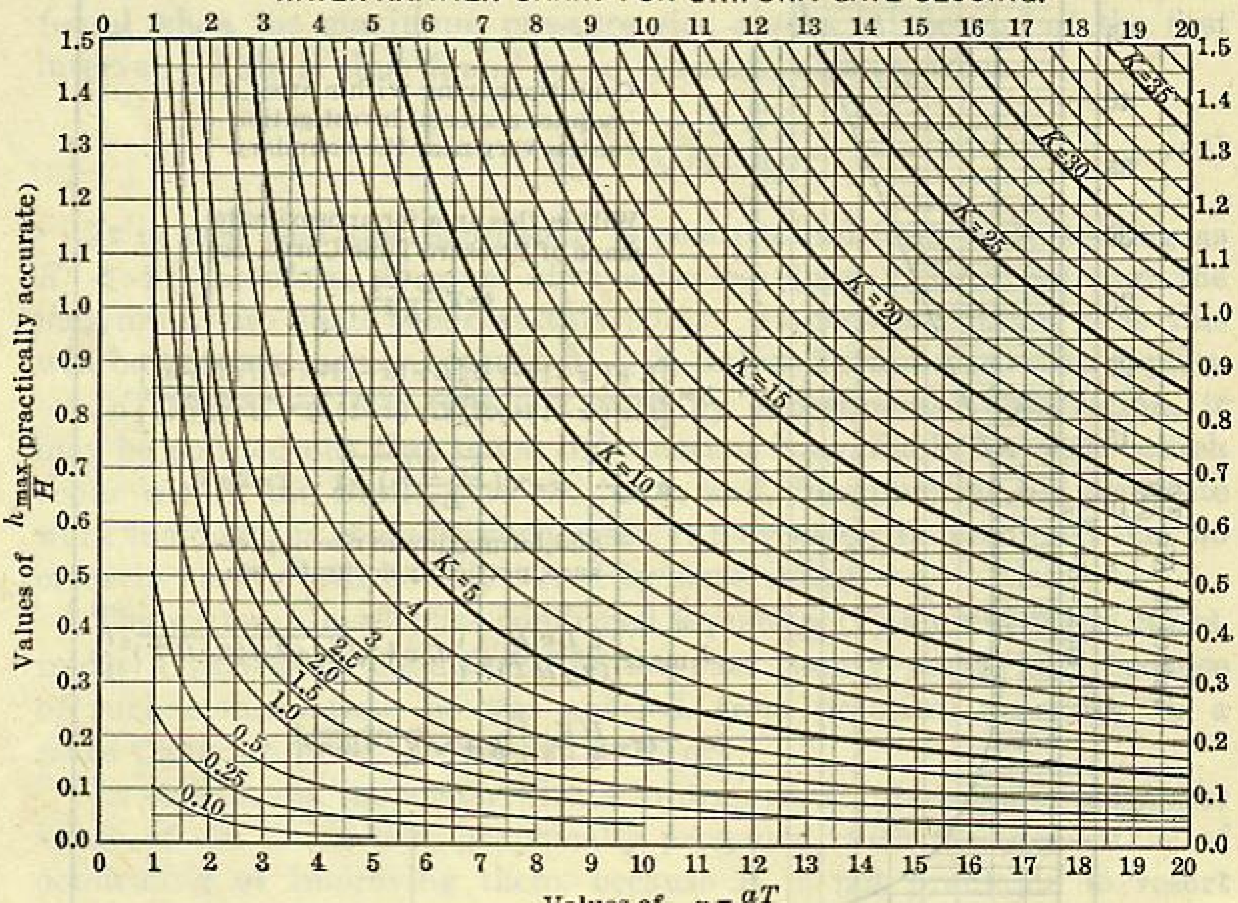

Fig. 13.

The effects of velocity head and friction head have been neglected in the preparation of Figs. 12 and 13. These factors are nearly always of minor importance, as affecting the rise of pressure, but, nevertheless, they may be taken into account by means of arithmetic integration in special cases where great accuracy is thought to be warranted. These effects, at any rate, cannot be included in a usable formula, covering the whole range of the charts.

For the benefit of those who are interested in the phenomena of water-hammer only sufficiently to desire a quick, comprehensive grasp of the derivation of the various formulas from fundamental principles, it seems worth while to present a simple analysis based on a straightforward hypothesis which has been generally accepted as applying to elastic columns, but which has not, so far as the writer knows, been directly and clearly stated. This is attempted in the following language: Mr. Johnson.

When a varying pressure is applied against the end of an elastic column, the change of velocity of that end, in the time required for the pressure wave to traverse the length of the column and return, is proportional to the arithmetic mean of the two pressures which there exist at the two extremities of this "interval" of time; and the impulse produced by the product of this average pressure and the time, $\frac{2L}{a}$, is equal to a change in momentum corresponding to the total mass of the column multiplied by the change in velocity of its end, adjacent to the point of application of the pressure.

This is a perfectly general statement applicable to any "interval," $\frac{2L}{a}$, of time during the acceleration (negative) of the elastic column.

If time is measured in "intervals" instead of seconds, we may write $\frac{2L}{a} \times n$ (intervals) $= t$ and $\frac{2L}{a} \times N = T$, where N is the total number of intervals (either a whole or mixed number) required to bring the end of the elastic column to a state of rest, and n, also either a whole or a mixed number, corresponding to any number of seconds, t.

With particular reference to a column of water arrested by the closure of a gate, we may designate the instantaneous pressures (h) above normal (H_0), at the extremities of any interval of time during closure of the gate, as h_{n-1} and h_n, respectively, and may write the general expression corresponding to the hypothesis (omitting the area of the pipe, assumed constant, and cancelling the common factor denoting the specific weight of the fluid), as follows:

$$\frac{h_{n-1} + h_n}{2} \times \frac{2L}{a} = \frac{L}{g}(V_{n-1} - V_n)$$

or, more simply,

$$h_{n-1} + h_n = \frac{a}{g}(V_{n-1} - V_n) \ldots\ldots\ldots\ldots\ldots (1)$$

This is the simple relation of which Mr. Gibson took advantage in working out his arithmetic tables, and to which he refers at the end of the paragraph on page 719.

During the first interval, V_{n-1} becomes the initial velocity, V_0, and h_{n-1} becomes $h_0 = 0$, so, for values of n equal to or less than unity (or for values of $t \overline{\overline{<}} \frac{2L}{a}$), we may write:

$$h_{n \overline{\overline{<}} 1} = \frac{a}{g}(V_0 - V_n) \ldots\ldots\ldots\ldots\ldots\ldots (1A)$$

Mr. Johnson.

If the gate is completely closed in one interval or less, we have the maximum possible rise of pressure; for, in that case, V_n becomes zero and we may write:

$$h_{max.} = \frac{a}{g} V_0$$

which is the famous formula of Joukovsky.

Thus, it is clear how all these fundamental equations are derivable from the one general hypothesis which, itself, reverts to the second law of Newton.

Equation (1) comprises the essential features of the water-hammer problem for fast or slow-closing gates, and a particular case may be worked out for any known or assumed character of closure, with respect to time, by expressing the simple relation between the quantity of water discharging at any time through the gate-opening existing at that time, and the total pressure head which drives the water through it. For example, if ϕ_n represents the proportion, at any time, of full gate-opening, the area of which is b, and the coefficient of discharge is c, we have (neglecting heads due to velocity and friction),

$$\phi_n\, b\, c \sqrt{2\, g\, (H_0 + h_n)} = A\, V_n \ldots\ldots\ldots\ldots\ldots (2)$$

and

$$\phi_0\, b\, c \sqrt{2\, g\, (H_0 + h_0)} = A\, V_0 = b\, c \sqrt{2\, g\, H_0} \ldots\ldots\ldots (3)$$

where A is the area (constant) of the pipe.

Dividing Equation (2) by Equation (3),

$$\phi_n\, V_0 \sqrt{1 + \frac{h_n}{H_0}} = V_n$$

or, putting $\frac{h_n}{H_0} = P_n$, we have,

$$V_n = \phi_n\, V_0 \sqrt{1 + P_n} \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

and

$$V_{n-1} = \phi_{n-1}\, V_0 \sqrt{1 + P_{n-1}} \ldots\ldots\ldots\ldots\ldots (5)$$

Subtracting Equation (4) from Equation (5),

$$V_{n-1} - V_n = V_0 \left(\phi_{n-1} \sqrt{1 + P_{n-1}} - \phi_n \sqrt{1 + P_n}\right) \ldots (6)$$

Thus, we have another expression proportional to ΔV, which may easily be made simultaneous with Equation (1), which latter may be rewritten, as follows:

$$V_{n-1} - V_n = \frac{g\, H_0}{a} (P_{n-1} + P_n) \ldots\ldots\ldots\ldots (7)$$

Equating the second terms of Equations (6) and (7), we have the general water-hammer formula for closing gates, as follows:

$$P_{n-1} + P_n = K\left(\phi_{n-1} \sqrt{1 + P_{n-1}} - \phi_n \sqrt{1 + P_n}\right) \ldots (8)$$

where $K = \frac{a\, V_0}{g\, H_0}$.

The successive values of ϕ with respect to time are quite independent of the other factors of the equation, and they may be all predetermined from an arbitrary or assumed character of gate closure, with respect to time. This may be expressed in the form of an independent curve plotted, for example, from some known motion of a particular gate with respect to time. Mr. Johnson.

If the gate closure is uniform, as Mr. Gibson has assumed in deriving his formulas, then the equation for ϕ may be written out, for subsequent use, or plotted as a straight line on a piece of cross-section paper. In this special case, the equation would be,

$$\phi_n = 1 - \frac{n}{N}, \text{ or } = 1 - \frac{t}{T}$$

as he states.

During the first interval, Equation (8) may be simplified to read,

$$P_{n=1} = K\left(1 - \phi_n \sqrt{1 + P_n}\right) \ldots\ldots\ldots\ldots\ldots (9)$$

In order to "get a start" in computing the successive values of P, one interval apart, the first value of $\phi_{n=1}$ may be selected from the gate-opening curve at any convenient proportion of the first interval, and $P_{n=1}$ found from Equation (9).

Then, and thereafter, the successive values of ϕ taken from the curve, one interval apart, may be substituted in Equation (8) and the successive values of P_n found from the known values of P_{n-1} as the computations progress. If the complete pressure-time curve is desired, several different proportions of the first interval must be selected and the corresponding values of ϕ introduced first in Equation (9).

Equations (8) and (9) may be rearranged, of course, in the form of the solution of quadratics, separating out the values of P_n on one side, as Mr. Gibson has done, or these values may be solved by trial as he has done in making up his tables. The result is, naturally, the same in either case, and one method is no more nor less accurate than the other.

It is interesting to note the physical story told by a slight rearrangement of Equation (1) when $2h_{n-1}$ is subtracted from both sides. It then appears in this form:

$$h_n - h_{n-1} = \frac{a}{g}\left(V_{n-1} - V_n\right) - 2\,h_{n-1} \ldots\ldots\ldots (10)$$

Now, the term $\frac{a}{g}\left(V_{n-1} - V_n\right)$ represents the "natural rise" in pressure in a pipe of indefinite length, or at least so long that returning waves do not interfere with the natural rise during the time taken for the velocity to change the stated amount. This is the case in the first interval as shown by Equation (1*A*).

We know from the principles laid down by Joukovsky that a supernormal pressure wave initiated at the beginning of any "interval" will become subnormal upon its return at the end of the "interval" of time, Mr. Johnson. or will be, so to speak, turned "upside down", and hence that its reducing effect on the pressure which would otherwise exist at the end of such "interval" is twice as great as the original supernormal value. This accounts for the term $2h_{n-1}$, and we may write another general hypothesis which is adequate for the statement of Equation (1), as follows:

The rise of pressure during any interval, $\frac{2\,L}{a}$, of time is equal to the "natural" or undisturbed rise, reduced by twice the value of the total pressure (above normal) which existed at the beginning of such "interval".

There are probably many ways in which to express the philosophy of Equation (1), and each may select the one best suited to his mind. The following is one of them:

If the pressure, h_{n-1}, should remain constant throughout the "interval" which it precedes, it is known, from the nature of an elastic column, that each and every particle of water would suffer the same diminution in velocity during the ensuing whole "interval" and, hence, that the change in velocity of the column, as a whole, would be faithfully recorded by the equal change at the end of the column, next to the gate. Call this portion of the velocity change, $\Delta_1 V$. Then, from the relation of impulse to change of momentum, we may write,

$$h_{n-1}\left(\frac{2\,L}{a}\right) = \left(\frac{L}{g}\right)\Delta_1 V\text{, or, } 2\,h_{n-1} = \left(\frac{a}{g}\right)\Delta_1 V$$

Now, any pressure added later in the interval under consideration, could have time, during such interval, to distort only a portion of the water column (of total round-trip length, $2L$), and the change in velocity produced at the end of the column by any such increment of pressure, Δh, may be derived as follows:

The mass affected in the time, dt, is proportional to adt.

This mass, multiplied by its acceleration, equals the increment of force to produce the acceleration. Therefore,

$$dh = \left(\frac{a}{g}\right) dt \times \left(\frac{dv}{dt}\right)\text{, or, } \Delta h = \left(\frac{a}{g}\right)\Delta V$$

Calling this additional change in velocity (of the end of the column), $\Delta_2 V$, we may write:

$$\Delta h = \left(\frac{a}{g}\right)\Delta_2 V$$

but,

$$V_{n-1} - V_n = \Delta_1 V + \Delta_2 V\text{, and, } \Delta h = h_n - h_{n-1}$$

Hence,

$$2\,h_{n-1} + h_n - h_{n-1} = h_n + h_{n-1} = \left(\frac{a}{g}\right)(V_{n-1} - V_n)$$

MINTON M. WARREN,* ASSOC. M. AM. SOC. C. E. (by letter).—This paper goes into the mathematical theory of ordinary water-hammer more thoroughly than any previous discussion of the question, and is certainly a valuable addition to the literature on this subject. The curves presented by the author show the differences in the various formulas with great clearness. Mr. Warren

The writer, however, does not believe that Mr. Gibson's method and formulas are practical for ordinary use for three reasons:

First.—In order to use the formula, a large amount of very tedious figuring must be done, and unless it is very carefully checked by logarithms, a very small arithmetical mistake may make a large error in the result.

Second.—The formulas have not been confirmed by experiments and are based on certain assumptions the accuracy of which are open to question.

Third.—In most cases, simpler formulas will give as accurate results as the data warrant.

It is almost impossible to get away from the idea that the more calculations and exact mathematical methods used in obtaining a result in engineering, the more accurate the result will be, regardless of the assumptions on which the calculation is based. It is the same false accuracy that leads engineers to submit cost estimates figured down to odd cents on projects running into millions.

The writer does not believe that the many assumptions needed in deducing any formula for slow-closing gates warrant the elaborate methods used by Mr. Gibson, until these methods and formulas are backed up by careful and extensive experimental data, as were the formulas of Professor Joukovsky.

In arriving at his results, Mr. Gibson assumes that the area of the gate is closed at a uniform rate, whereas the writer, in deducing his simple formula,

$$h = \frac{L\,V}{g\left(T - \frac{L}{a}\right)}$$

assumed that the gate moved in such a way as to cause the pressure to rise at a constant rate. In any given case, neither assumption is strictly true, and experiment alone will show which is nearer the average gate motion.

Referring to Fig. 9, it is seen that, according to the author's calculations, such a gate motion, giving a constant rise of pressure, is not far from a straight line, and probably as near the truth in ordinary

* Boston, Mass.

Mr. Warren. gates as his assumption (uniform reduction of area), which makes the mathematics very much more difficult.

There is great need for a series of careful experiments on slow-closing gates, and it is to be hoped that Mr. Gibson, or some other member of the Society, will undertake this work. Data from such tests properly used would be of more service to the Profession than the most perfect theoretical formulas which have never been tested. Until this is done, engineers will have to base design on the meager practical data available, guiding their judgment by unproved formulas, and the writer has found that the simple formula given above comes nearer the pressures he has observed in practice than other formulas, although, like the others, it is based on certain assumptions which are not strictly true.

In recent tests by one of the best-known manufacturers of water-wheel governors, the author's formula has given results so closely in accord with the tests, that it has been adopted for general use. As this company has many chances to obtain practical operating values of water-hammer, its conclusions are certainly of interest, in default of other experimental data.

In regard to Mr. Halmos' statement that this formula can never give true results, even accidentally, it should be noted that, under certain conditions, it gives the same results as the formula which he uses (Allievi's), and, in some of the tests mentioned in the previous paragraph, has given results within the probable error of the pressure gauges used.

In one point, Mr. Gibson's formula gives what appears to the writer unreasonable results. This is illustrated in Fig. 4, where the curve rises very sharply between heads of from 10 to 100 ft. Alliévi's curve rises even more sharply and can be easily proved wrong, as it reaches values of h which are far above the maximum value possibly reached in instantaneous closing.

Mr. Gibson's curve stops at that value, but the writer does not believe that experiments confirm the large increase in water-hammer for low heads over that for high heads. His assumption that the waves are perfectly reflected from the slowly closing gate is also open to question, and was not proved in Joukovsky's experiments.

It is not intended to imply that Mr. Gibson's formulas may not prove to be as accurate or even more accurate than the others, but, in the absence of any experimental proof, their added complexity does not seem to be warranted, in view of the approximations and assumptions on which they are based.

Mr. Anderson. J. T. NOBLE ANDERSON,* M. AM. SOC. C. E. (by letter).—On several occasions, the writer has investigated this question, or a cognate phe-

* Narbethong, Victoria, Australia.

nomenon, with pumps—centrifugal and others—through long pipes. In one case, there were two pipes with a common penstock—each of wood, 3 ft. in diameter, and 2 300 feet long. In every case, he has been greatly impressed by the extreme complexity of the subject. Mr. Anderson.

With the help of Professor Joukovsky's theory and some "rule-of-thumb" guesses, gained from a general experience, these pipe lines and the values concerned can now be calculated to as close an economic margin as most other engineering problems. At the same time, the result of experience shows so many baffling discrepancies, that the experimental results, which the author hopes to obtain "in the not too distant future", will be waited for eagerly.

The care he has taken to collect and co-ordinate the formulas, is an earnest that he may be relied on to record fairly and fully not only his results, but all the extraneous facts and happenings which may, however remotely, bear on the many irregularities between actual records and what the formulas anticipate. In particular, it might help if two slightly disturbing factors were recorded in each test, that is: (1) the temperature of the water, and (2) the proportion of free air present in the water.

The second factor will probably be quite immaterial, but, in a case where the substance pumped through a centrifugal was sewage with a specific gravity of 64 lb. per cu. ft., the writer found that the gases contained had a very disturbing influence.

In some recent investigations, the writer used a steam indicator with springs specially chosen for the anticipated pressure. Some indication of the author's apparatus will no doubt be given.

Ford Kurtz,* Esq.† (by letter).—Mr. Gibson has made a valuable addition to the too few and scanty English treatises on the mathematical theory of water-hammer. His treatment of the subject, however, is chiefly of value in obtaining, without the use of differential equations and from physical laws the import of which is readily grasped, formulas which give the same practical results as the much simpler and less cumbersome equations of Lorenzo Alliévi, first published in Rome, in 1903. These equations must not be confused with the confessedly approximate and inadequate formula designated by Mr. Gibson as the "Alliévi formula", and on which Mr. R. D. Johnson has apparently founded a pressure-time equation. The equations referred to are mathematically rigid formulas which take into account not only the effect of net head but also the compressibility of the water and the extensibility of the pipe and which, as far as the writer knows, have never before been published in English. In 1911, the writer prepared, for his own use, a translation of a German translation of Mr. Alliévi's Mr. Kurtz.

* New York City.

† Now M. Am. Soc. C. E.

Mr. Kurtz. work, and found the latter's treatment so remarkably comprehensive and thorough that he has used it ever since in all water-hammer problems. The German translation* can be found in the Engineering Societies Library.

In presenting the exact formulas of Alliévi, the following nomenclature will be added to that of Mr. Gibson's paper:

$R_0 = \frac{a\,V_0}{g}$ = excess, or water-hammer, head due to instantaneous complete closure of gate.

$m = \frac{a\,V_0}{g\,H_0}$ = ratio of instantaneous water-hammer head to net head.

$F(t)$ and $f(t)$, or simply F and f = certain functions of time, t.

$z = \frac{H_0 + h_t}{H_0}$ = ratio of total variable head to net head.

$\phi(t)$ = gate-opening at time, t, as a ratio of maximum gate-opening.

The exact formulas of Alliévi contain a term which makes it possible to determine the pressure at any point of the pipe line at any moment, but the writer is presenting only the simple form for determining pressure at the outlet, or discharge section just up stream from the gate, as that is the problem investigated by Mr. Gibson. It is also assumed that the pipe line is of uniform thickness and diameter throughout its length. Then, during the period,

$$O < t \leqq \frac{2\,L}{a},$$

$$z = 1 + \frac{F(t)}{H_0},$$

where

$$\frac{F(t)}{H_0} = m + \frac{m^2}{2}[\phi(t)]^2 - m\,\phi(t)\sqrt{1 + m + \frac{m^2}{4}[\phi(t)]^2}$$

and, during the period,

$$\frac{2\,L}{a} < t \leqq T,$$

$$z = 1 + \frac{F(t)}{H_0} - \frac{F\left(t - \frac{2\,L}{a}\right)}{H_0} = 1 + \frac{F(t)}{H_0} - \frac{f(t)}{H_0},$$

* "Allgemeine Theorie über die veränderliche Bewegung des Wassers in Leitungen," von Lorenzo Alliévi, 1909.

Mr. Kurtz.

where

$$\frac{F(t)}{H_0} = m - \frac{f(t)}{H_0} + \frac{m^2}{2}[\phi(t)]^2 - m\,\phi(t)\sqrt{1 + m - \frac{2\,f(t)}{H_0} + \frac{m^2}{4}[\phi(t)]^2}$$

For the linear law of gate movement, $\phi(t) = 1 - \frac{t}{T}$, as already stated by Mr. Gibson.

These formulas applied to Mr. Gibson's first example give the following equations:

For $O < t \leqq 0.35$ sec.,

$$\frac{F(t)}{H_0} = 10.35008 + 53.56185\,\phi^2 - 10.35008\sqrt{11.35008\,\phi^2 + 26.78092\,\phi^4}$$

and, for $0.35 < t \leqq 6.0$ sec.,

$$\frac{F(t)}{H_0} = 10.35008 - \frac{f(t)}{H_0} + 53.56185\,\phi^2 - 10.35008\sqrt{\left(11.35008 - \frac{2\,f(t)}{H_0}\right)\phi^2 + 26.78092\,\phi^4}$$

As already noted by Mr. Gibson, the values of $\frac{F(t)}{H_0}$ and $\frac{f(t)}{H_0}$ are so small compared with some of the individual terms of the equations, that it is necessary to use logarithms in solving the equations.

Using the exact Alliévi equations as given previously, Table 6 has been prepared, showing the rise of pressure and also the differences between the values of the rise obtained by Alliévi equations and those obtained by Mr. Gibson's equations. The maximum divergence of less than one-half of 1% shows the remarkable agreement of the two methods. It is apparent, however, that the simplicity of the solution by the Alliévi formulas, without giving heed to magnitude and direction of waves in the computations, makes it far superior to that of Mr. Gibson as a working method.

The exact formulas of Alliévi also furnish equations similar to those given previously for the case of the opening of a valve at the lower end of a pipe line, either from fully closed position, or from some initial partial opening. As already stated, in their complete form they also give the pressure at any time for any point along the pipe line, thus covering the matter mentioned by Mr. Gibson as to be discussed by O. V. Kruse, Assoc. M. Am. Soc. C. E. Of course, there are also exact equations for the velocity at any time for any point along the pipe line.

Mr. Kurtz.

The partial differential equations for the general motion of water in pipes are based on the fundamental differential formulas for the motion of water in general. Unfortunately, these partial differential equations, four in number, cannot be integrated (not even by approximate arithmetic integration so far as the writer knows) without making the following simplifying approximations:

1. Velocity in direction of axis of pipe considered uniform over any chosen section of the stream.

2. Skin friction and viscosity neglected.

3. Velocities at right angles to the axis of the pipe, due to expansion or contraction of the pipe by changes in pressure, neglected.

4. Pressure considered uniform over any chosen section of the stream.

5. Assumed that the pipe consists of individual circular elements independent of each other, which are freely extensible.

6. Assumed that the ratio of velocity of water in the pipe to the velocity of propagation of pressure changes is small enough, compared with unity, so that its addition thereto or subtraction therefrom can be neglected in every case at every instant.

By making these approximations, we obtain the so-called exact formulas of Alliévi.

Mr. Gibson's method of taking account of skin friction (which is only approximate, as has been pointed out by William P. Creager, M. Am. Soc. C. E.) could easily be applied to the exact Alliévi formulas by changing the factor m so as to have it correspond at all times to $(H_0 + h_f)$ instead of to H_0. The exact Alliévi formulas can be applied so as to take account of varying diameters and thicknesses of pipe in the same line, but they soon lead to so much complication that they become impracticable. In such cases, the writer uses the formulas as already given for a pipe of uniform diameter and thickness, but gives to m the value, $\dfrac{Q_0 \, \Sigma \dfrac{L}{A}}{g \, H_0 \, \Sigma \dfrac{L}{a}}$, where Q_0 equals the flow of water in a pipe at full gate-opening, L equals the length of any section of pipe, A, its cross-sectional area, and a, its individual value of the velocity of propagation of pressure changes. Also, the factor, $\dfrac{2\,L}{a}$, must everywhere be changed to read $2\,\Sigma\dfrac{L}{a}$. This is confessedly an approximation made without mathematical proof, but it is probably exact enough for practical purposes in the majority of problems.

Mr. Kurtz.

TABLE 6.

Interval in terms of $\frac{2L}{a}$	$\phi(t)$	$\frac{F(t)}{H_0}$	$\frac{f(t)}{H_0} = \frac{F\left(t - \frac{2L}{a}\right)}{H_0}$	$z - 1$	h_t, Alliévi	h_t, Gibson	Percentage of divergence.
0	1.00000	0.00000	....	0.00000	0.00	0.00	.....
¼	0.95833	0.07338	....	0.07338	12.10	12.12	0.16
½	0.91667	0.15476	....	0.15476	25.53	25.53	0.00
¾	0.875	0.24503	....	0.24503	40.43	40.41	0.05
1	0.83333	0.34550	0.00000	0.34550	57.01	56.96	0.09
¼	0.79167	0.49780	0.07338	0.42442	70.03	69.94	0.13
½	0.75	0.66286	0.15476	0.50810	83.84	83.72	0.14
¾	0.70833	0.84160	0.24503	0.59657	98.43	98.31	0.12
2	0.66667	1.03529	0.34550	0.68979	113.82	113.63	0.17
¼	0.625	1.26152	0.49780	0.76372	126.01	125.78	0.18
½	0.58333	1.50125	0.66286	0.83839	138.33	138.05	0.20
¾	0.54167	1.75454	0.84160	0.91294	150.64	150.32	0.21
3	0.5	2.02152	1.03529	0.98623	162.73	162.42	0.19
¼	0.45833	2.30592	1.26152	1.04440	172.33	171.97	0.21
½	0.41667	2.60055	1.50125	1.09930	181.38	181.10	0.15
¾	0.375	2.90456	1.75454	1.15002	189.75	189.43	0.17
4	0.33333	3.21686	2.02152	1.19534	197.23	196.77	0.23
¼	0.29167	3.53609	2.30592	1.23017	202.98	202.49	0.24
½	0.25	3.86010	2.60055	1.25955	207.83	207.14	0.33
¾	0.20833	4.18755	2.90456	1.28299	211.69	211.00	0.33
5	0.16667	4.51715	3.21686	1.30029	214.55	213.79	0.35
¼	0.125	4.84719	3.53609	1.31110	216.33	215.75	0.26
½	0.08333	5.17707	3.86010	1.31697	217.30	216.71	0.26
¾	0.04167	5.50630	4.18755	1.31875	217.59	216.95	0.28
6	0.00000	5.83293	4.51715	1.31578	217.10	216.57	0.23
Average percentage....	....	....	....	....	....	....	0.20

WILLIAM P. CREAGER,* M. AM. SOC. C. E. (by letter).—The theory of water-hammer in penstocks is one of the most intricate problems confronting engineers. Mr. Gibson has made a considerable addition to the knowledge of the subject; but exact solutions for all conditions have not yet been reached. Mr. Creager.

The author has developed rational equations for penstocks of constant diameter and negligible friction head. In cases where friction head is relatively large, and particularly where the penstock has a varying diameter, we are still very far from a practical solution.

At any instant during gate closure, the discharge through the gate is a function of the static head plus water-hammer head less friction head, all measured at the gate.

On page 725, the author makes the assumption that the friction head at the gate is proportional to the square of the velocity adjacent to the gate. This assumption appears to the writer to be only approximate, since it is well known that, at any instant during surges, the velocity is materially different at different points on the penstock.

* New York City.

Mr. Creager. It is the writer's opinion that, during the period, $\frac{2L}{a}$, subsequent to a single small instantaneous closure, the friction head at the gate is not constant, although the velocity adjacent to the gate during that period is constant. This constantly varying friction head makes it very difficult to include this feature in the equations for water-hammer and, in all probability, the author's methods are as close as can be obtained. It would be of interest to know how much difference the inclusion of friction head makes in ordinary problems.

For a penstock with varying diameter, auxiliary waves are set up each time a water-hammer wave passes a point of change in diameter. For this condition, the author's equations would not apply. His methods would apply, but they would be exceedingly difficult, if not impossible, of practical application.

It is evident that the maximum rise of pressure is materially influenced by the characteristics of the gate-closure curve. It is well known that modern turbine governors do not provide a uniform rate of gate closure throughout the stroke. It is also possible that governors of different types have different closure characteristics. Consequently, complete data for the determination of maximum water-hammer for specific cases must include the gate-closure characteristics of the governor which is to be a part of the machinery.

Mr. Gibson. NORMAN R. GIBSON,* M. AM. SOC. C. E. (by letter).—In closing the discussion of his paper, the writer desires, first of all, to make amends, as far as possible, for the injustice unknowingly done to Mr. L. Allievi by the reference to his formula on the diagrams and in the text of the paper. As pointed out by Mr. Halmos, the formula designated by the writer as Allievi's is only one of several contained in his complete work. While "ignorance excuseth none", the writer can only plead, in extenuation, that he did not read Allievi's work, because he could not find an English translation of it, but was led to believe, by the frequent references to "Allievi's formula" in English literature on the subject, that the formula quoted was, in fact, the one and only equation derived by Allievi. The writer was the more completely led astray by a lengthy reference to this formula in a paper published in the *Transactions* of the Society, from which it could not but be inferred that the author had read the work of Allievi in the original language. It is inconceivable how any one, having read it, could so overlook the most important part of Allievi's work.

The writer is indebted to Mr. Halmos and Mr. Kurtz for having brought to light some further formulas of Allievi, as published in the German and Italian languages. Mr. Halmos states that the writer's solution agrees in every particular with that of Allievi, and Mr. Kurtz finds a close agreement in an arithmetical example.

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* Niagara Falls, N. Y.

The identity of the results obtained by the Allievi equations with those derived from the writer's may be proved as follows: Mr. Gibson.

Considering first Equation (3) presented by Mr. Halmos on page 750, it should be noted that Mr. Halmos had the writer's attention drawn to a mistake in his definition of $\psi(t)$. This symbol stands for the ratio of the instantaneous area of discharge to the area of the pipe (not the area of the gate-opening at the time, $t = 0$).

The two nomenclatures are as follows:

Halmos. Gibson.

$$\psi(t) = \left(1 - \frac{t_n}{T}\right) \frac{B_0}{\sqrt{2\,g}} = \sqrt{\frac{S_{t_n}\,g}{2\,a^2}}$$

$$y_0 = H_0$$

$$y = H_0 + h_{t_n}$$

$$H = H_0 + \frac{a\,V_0}{g}$$

$$c_0 = V_0$$

$$C = V$$

On page 750, Mr. Halmos states that $F(t)$, found in the first interval, should be substituted for (f) in Equation (3), in order to obtain the value of y for every subdivision of the second interval.

Using the subscript n of the writer's notation, the relation between $F(t)$ and (f), may be expressed by the equation:

$$f(t_n) = F(t_{n-1})$$

Again, on page 750, Mr. Halmos gives:

$$F(t) = \frac{a}{g}(c_0 - C) - f$$

Substituting the subscript, n, and the writer's notation for c_0 and C,

$$F(t_n) + f(t_n) = \frac{a}{g}(V_0 - V_{t_n})$$

$$\therefore 2\frac{a}{g}(V_0 - V_{t_n}) = 2\{F(t_n) + F(t_{n-1})\} \text{..........} (1)$$

and, similarly,

$$2\frac{a}{g}(V_0 - V_{t_{n-1}}) = 2\{F(t_{n-1}) + F(t_{n-2})\} \text{........} (2)$$

Subtracting Equation (1) from Equation (2):

$$2\frac{a}{g}(V_{t_n} - V_{t_{n-1}}) = 2\{F(t_{n-2}) - F(t_n)\}$$

If each term in the writer's general expression for R is now expanded in terms of V, then:

$$R_{t_n} = R_0 + 2\frac{a}{g}\{(V_{t_n} - V_{t_{n-1}}) + (V_{t_{n-2}} - V_{t_{n-3}}) + \text{..............}\}$$

Mr. Gibson.

and, by analogy:

$$R_{t_n} = R_0 + 2 \left[\left\{ F(t_{n-2}) - F(t_n) \right\} + \left\{ F(t_{n-4}) - F(t_{n-2}) \right\} + \ldots \right]$$

and, by cancellation:

$$R_{t_n} = R_0 - 2\, F(t_n)$$

and

$$R_{t_{n-1}} = R_0 - 2\, f(t_n)$$

By substitution of the above values and by reference to the relation of terms in the paper, Allievi's Equation (3), as given by Mr. Halmos on page 750, may be written for any interval, n:

$$(H_0 + h_{t_n})^2 - 2\,(H_0 + h_{t_n}) \left(H_0 + R_{t_{n-1}} + \frac{S_{t_n}}{2} \right) + \left\{ H_0 + R_{t_{n-1}} \right\}^2 = 0$$

and by solving this quadratic:

$$h_{t_n} = \frac{1}{2} \left\{ (S_{t_n} + 2\, R_{t_{n-1}}) \pm \sqrt{St_n\,(St_n + 4\, R_{t_{n-1}} + 4\, H_0)} \right\} \ldots (3)$$

which is the writer's general expression, similar to Equation (10) on page 724, for h at any time in any interval.

Mr. Kurtz has presented the general Allievi formula in somewhat different form from that used by Mr. Halmos, in that the values of V and h at the interval points, as contained in the writer's values of R, are eliminated, and the labor of computation is thereby reduced. The writer's equations, however, require no more heed to be given to "magnitude and direction of waves in the computations" than do the Allievi equations. Mr. Kurtz's remark in this connection, on page 767, probably refers to the explanation of the work of arithmetic integration which preceded the writer's derivation of his formulas. Mr. Kurtz's Table 6 of arithmetic errors indicates merely the difference in the results that might be obtained by two computers. The results given by the Allievi formulas are identical with those of the writer.

The elimination from the writer's equations of V and h at the interval points may be accomplished in the following manner: Referring to the equations on page 724, the expression for R_{t_1} may be equated to that part of the equation remaining after R_{t_1} has been substituted in place of $\frac{a}{g} V_1 + h_1 - C_{t_1}$.

Thus, in general,

$$R_{t_{n-1}} = \frac{a}{g} \left(1 - \frac{t_n}{T} \right) B_0 \sqrt{H_0 + h_{t_n}} + h_{t_n} = \frac{a}{g} V_{t_n} + h_{t_n}$$

$$= \frac{a}{g} V_n + \frac{a}{g} (V_n - V_{t_n}) + h_{n-1} - C_{t_{n-1}} = \frac{a}{g} V_{t_{n-1}} - h_{t_{n-1}}$$

$$= \frac{a}{g} \left(1 - \frac{t_{n-1}}{T} \right) B_0 \sqrt{H_0 + h_{t_{n-1}}} - h_{t_{n-1}}$$

$$\sqrt{S_{t_{n-1}}\,(H_0 - h_{t_{n-1}})} - h_{t_{n-1}} \ldots\ldots\ldots\ldots (4)$$

If this value of R_{n-1} is substituted in the writer's equations in place of the value given in the nomenclature on page 722, the labor of solving problems by the formulas will be lessened. Mr. Gibson.

Mr. Johnson's discussion contains considerable new matter which will be found most helpful in solving problems of the kind under consideration. His remarkably simple and concise presentation of the fundamental principles of this subject and his derivation of the expression for $\frac{h_n}{H_0}$, the ratio at any time of the rise of pressure to the net head, are particularly noteworthy and important. His chart of limitations, shown in Fig. 12, makes it possible to see at a glance where it is safe to apply the general logarithmic curve given by Allievi's Formula (4), which is Mr. Johnson's Equation (7), and also to see where the maximum pressure rise occurs at the end of the first interval. As pointed out by Mr. Johnson, his Equation (7) covers a wide range of conditions, and it will be found convenient to use this equation wherever it applies. In Fig. 13, Mr. Johnson has indicated how an exceedingly useful chart may be prepared by resorting to the expedient of solving for the ratio, $\frac{h_{max.}}{H_0}$, instead of for $h_{max.}$ directly. Such a chart may be made to show graphically a range of results as wide as desired and, as stated in the discussion, each point on the chart covers an infinite number of cases.

Mr. Creager's desire to know more about the application of water-hammer formulas to pipes of varying diameter will be appreciated by every one who studies this subject. Needless to say the problem is a complex one, but it seems certain that the fundamental relations that have been developed herein can be applied not only to this but to any water-hammer problem. The point brought out by Mr. Creager that the friction head at the gate is not constant during the period of $\frac{L}{a}$ is worthy of consideration, but the writer would suggest that the change in friction head, being a change in pressure, cannot traverse the length of the pipe at a rate faster than the velocity, a. In any event, even though the friction head does vary during the interval periods, it would seem that the net result throughout the closing time would be that the friction and velocity heads are recovered proportionately to the rate at which the velocity of flow is destroyed.

On page 769, Mr. Creager misquotes the assumption made on page 725 of the paper. The friction head at the gate was assumed to be proportional to the square of the velocity of flow, not the velocity "adjacent to the gate."

During the past year many experiments on pressure rise have been made by the writer with specially designed apparatus. Most of the ex-

Mr. Gibson. periments, however, were made in connection with his new process for the measurement of the flow of fluids in closed conduits, and as this work continued until late autumn, it has been necessary to postpone until later the particular experiments that have been proposed to show the relation between gate motion and pressure rise covered by the formulas. As soon as possible the results of these experiments will be made available for publication, and in assembling the data Mr. Anderson's request for full particulars will be remembered.

In reply to the criticism of Mr. Warren, it may be observed at once that the vital importance of including the initial static head among the factors that determine the maximum rise of pressure, cannot be considered in the same class as "odd cents in million dollar estimates". The large increase in water-hammer for low heads over that for high heads when the gate motion is uniform, is caused by the influence of the initial head.

Mr. Warren cannot obscure the points at issue by clouding the discussion with conjectures as to whether uniform gate motion or the gate motion he has assumed, but not defined, "is nearer the average gate motion". The vagaries of gate motions are not under consideration, as is clearly stated on page 710. The problem considered is the rise of pressure following any given gate closure, and, for this purpose, in deriving the formulas, uniform gate motion has been assumed. Diagrams of non-uniform gate motion were shown and attention was drawn to the effect of such non-uniform motion on the shape of the pressure-time diagram. In all practical cases, actual conditions will be determined and, as Mr. Johnson says, "the element of judgment must enter as usual". Mr. Warren's statement that his formula is based on a certain prescribed shape of the pressure-time diagram which may be produced by a special, though undescribed, performance of the gate, condemns his formula for general use, because it is impossible to adapt it to any particular gate motion. Indeed, what he says of his own arbitrary diagram is equally true of any form whatever which gives a value of the maximum pressure rise greater than the mean effective pressure rise and which encloses a proper impulse area.

For uniform closure, Mr. Warren's formula always leads to errors on the wrong side, that is, it gives less than the true value. In the region, pointed out in the paper, where it is approximately correct, and when the first interval is a small proportion of the closing time, as it frequently is in ordinary cases, this formula gives a result close to the value of the mean rise of pressure which can be determined by the simple expression, $\frac{L\,V}{g\,T}$.

Mr. Warren questions the assumption that pressure waves are perfectly reflected from the slowly closing gate. The question implies a misconception of the nature of the waves themselves. They are not of a substance that rebounds like a rubber ball, but are conditions of the water column at various times at different points along the pipe line. When a wave is started by a movement of the gate, the water is put into a state of super-normal pressure and it remains in this condition at the gate until the wave has traversed the length of the pipe to the origin and back again. When the wave has returned to the gate, the state of super-normal pressure there is converted to one of sub-normal pressure and the water there remains in this condition until the wave has again traversed the distance to the origin and back, whereupon the condition at the gate again changes to a state of super-normal pressure, and so on. At points along the pipe the changes of condition occur in the same sequence, except that a state of normal pressure exists for a time between the periods of super-normal and sub-normal pressures. When the gate is gradually closing, the algebraic sum of the innumerable infinitesimal waves (super-normal pressure being considered positive and sub-normal pressure negative) fixes the pressure that exists at the gate at any time during the closure. Mr. Gibson.

The authority for the assumption that the flow of water in the pipe does not affect the wave is to be found on page 709 in the following quotation from Miss Simin's translation of Joukovsky's work:

> "If the water column continues flowing, such flow exerts no noticeable influence upon the shock pressure. In a pipe from which water is flowing, the pressure wave is reflected from the open end of the pipe, in the same way as from a reservoir with constant pressure."

Finally, it may be stated definitely that, as an empirical rule for convenient use, Mr. Warren's formula is quite untrustworthy, and as an equation showing the relation between velocity destroyed and rise of pressure, it is fundamentally and inherently wrong.